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Abstract

Helically symmetric hydromagnetic equilibria with small helical corrugation are considered in a finite cylindrical domain. These solutions may be regarded as bifurcated from a circularly symmetric equilibrium with the plasma current at the plasma surface. Stable bifurcated equilibria must have a bifurcation point in the neighborhood of the margin of stability for the bifurcating equilibria. Using the energy principle we show that there are stable helical equilibria with long helical wavelength, the corresponding circular equilibria are unstable.

ABSTRACT

Helically symmetric hydromagnetic equilibria with small helical corrugation are considered in a finite cylindrical domain. These solutions may be regarded as bifurcation from a circularly symmetric equilibrium with the plasma enclosed by the plasma surface. Stable bifurcated equilibria must have a bifurcation point in the neighborhood of the margin of stability for the bifurcating equilibria. Using the energy principle we show that there are stable helical equilibria with long helical wavelength, the corresponding circular equilibria are unstable.

I. Introduction

Bifurcation of solutions representing equilibria occurs in many physical problems.[1] A well-known example is the buckling of an elastic beam. As the characterizing parameters vary, one family of solutions (the bifurcating equilibria) changes from stable to unstable, and the other family of solutions (the bifurcated equilibria) branches out at the bifurcation point. The bifurcated equilibria are usually of lesser symmetry than the bifurcating equilibria. Since solutions representing stable hydromagnetic equilibria are important in magnetic confinement of a plasma, it is of interest to study the stability of bifurcated hydromagnetic equilibria.

The simplest hydromagnetic model for a closed system of plasma containment is a straight sheet pinch. Namely, inside a circular cylinder of conducting wall the plasma forms a column whose cross-section is a circle concentric with the outer wall. In the plasma region, the magnetic field has only an axial component, and in the vacuum region between the plasma and the conducting wall the magnetic field has both an azimuthal as well as an axial component. Besides the size (as specified by its length and radius) of the wall, under the assumption of a sharp boundary for the plasma-vacuum interface the system is characterized by four physical parameters: the axial magnetic flux in the plasma, the axial flux in the vacuum, the total azimuthal flux, and the volume of the plasma. Given the four parameters, in addition to the circular equilibrium helical equilibria are other possible configurations. The helical structure is specified

by an axial wavenumber, an azimuthal wavenumber, and the amplitude of helical corrugation. The admissible axial wavenumbers are determined by the length of the conducting wall. The amplitude of corrugation is determined by the physical parameters.

The bifurcation problem for a hydromagnetic system was first discussed by Friedrichs.[2] He considered the same helical equilibria as is discussed in this paper in the limit of large wavenumbers (both axial and azimuthal). Clearly, the helical equilibria with small amplitude can be stable only when the bifurcation point is in the neighborhood of the margin of stability for the corresponding circular equilibria. This requires the azimuthal wavenumber to be equal to unity. Also it is clear that any instability for such bifurcated equilibria must arise from disturbances with an azimuthal wavenumber equal to unity and an axial wavenumber differing at most slightly from that of the helical structure.

In Section II we consider helically symmetric equilibria, expanding the solution in a small parameter, the amplitude of helical corrugation of the plasma-vacuum interface. In Section III we discuss bifurcation points in a parameter space. Then, we study the stability of the bifurcated helical equilibria, using the energy principle[3] for hydromagnetic stability. In minimizing the second variation of the potential energy, the displacement is compressible if the disturbance is in the same mode as that of the helical structure, and is incompressible otherwise. We treat the two cases separately in Sections IV and V.

Finally in Section VI we show that there are stable helical equilibria with long helical wavelength.

Usually when the characterizing parameters are such that the circular equilibrium is unstable, the condition is interpreted as unstable. However, if there exists a stable helical equilibrium for the same parameters, then the correct interpretation should be that the condition is stable with the plasma appearing in a helical configuration.

Recently, hydromagnetic equilibria with helical symmetry were calculated numerically by Friedman.[4] And stability for certain equilibria with long helical wavelength was analyzed by Weitzner.[5]

II. Equilibria with Helical Symmetry

We consider a helically symmetric, hydromagnetic equilibrium, in which a plasma is confined by magnetic fields inside a circular cylinder of conducting wall. The plasma current is assumed to be concentrated at the surface and the sharp plasma-vacuum interface is assumed to have a small amplitude of helical corrugation.

In cylindrical coordinates (r,θ,z) the helical symmetry requires that all quantities depend only on the radial variable r and the helical variable

$$\phi \equiv m\theta + kz \ ,$$

where m and k are the azimuthal and axial wavenumbers characterizing the helical structure (m can be any integer but k must be one of the discrete values determined by the length of the wall). The magnetic field $\vec{B} \equiv (B_r, B_\theta, B_z)$ can be described in terms of a flux function $\Psi(r,\phi)$. The requirement $\nabla\cdot\vec{B} = 0$ is satisfied by the specification $B_r = -r^{-1}\partial\Psi/\partial\phi$ and $mB_\theta + krB_z = \partial\Psi/\partial r$. Since $\vec{B}\cdot\nabla\Psi = 0$, Ψ is constant on each flux surface. The assumption $\nabla\times\vec{B} = 0$ requires that Ψ satisfy

$$\text{(1)} \qquad \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{m^2+k^2r^2}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} = \frac{2mk\Lambda}{(m^2+k^2r^2)^2} \ ,$$

where Λ is a constant equal to $mB_z - krB_\theta$. Accordingly,

$$\text{(2)} \qquad \vec{B} = \left(-\frac{1}{r}\frac{\partial\Psi}{\partial\phi} \ , \ \frac{m\frac{\partial\Psi}{\partial r} - kr\Lambda}{m^2+k^2r^2} \ , \ \frac{kr\frac{\partial\Psi}{\partial r} + m\Lambda}{m^2+k^2r^2}\right) \ .$$

Thus, on a cylindrical surface r = const. the magnetic field has a component equal to $(m^2+k^2r^2)^{-1/2}\Lambda$ along the helix

$m\theta + kz = $ const. and a component equal to $(m^2+k^2r^2)^{-1/2}\partial\Psi/\partial r$ perpendicular to the helix. The boundary conditions to be satisfied are that the plasma-vacuum interface and the conducting wall are two flux surfaces and the jump in the magnetic pressure across the interface is balanced by the constant plasma pressure.

The helical configuration is characterized by four physical quantities:

the axial magnetic flux in the plasma region

$$A \equiv \int_{\theta=0}^{2\pi} \int_{r=0}^{\rho} B_z r \, dr \, d\theta \,, \tag{3}$$

the axial magnetic flux in the vacuum region

$$B \equiv \int_{\theta=0}^{2\pi} \int_{r=\rho}^{e} B_z r \, dr \, d\theta \,, \tag{4}$$

the total azimuthal magnetic flux

$$C \equiv \int_{z=0}^{L} \int_{r=0}^{e} B_\theta \, dr \, dz \,, \tag{5}$$

and the volume of the plasma

$$D \equiv \int_{z=0}^{L} \int_{\theta=0}^{2\pi} \frac{1}{2} \rho^2 \, d\theta \, dz \,, \tag{6}$$

where e and L are the radius and the length of the cylindrical wall, and ρ is the plasma-vacuum interface: $r - \rho(\phi) = 0$.

We shall use the following notations defined in terms of the standard modified Bessel functions of integral order $I_\nu(x)$,

$K_\nu(x)$ and their derivatives $I'_\nu(x)$, $K'_\nu(x)$:

$$\mathcal{I}(r;\nu,\kappa) \equiv \frac{1}{\kappa d} \frac{I_\nu(\kappa r)}{I'_\nu(\kappa d)} ,$$

$$\mathcal{K}(r;\nu,\kappa) \equiv \frac{1}{\kappa d} \frac{K_\nu(\kappa r) I'_\nu(\kappa e) - I_\nu(\kappa r) K'_\nu(\kappa e)}{K'_\nu(\kappa d) I'_\nu(\kappa e) - I'_\nu(\kappa d) K'_\nu(\kappa e)} ,$$

$$W(\nu,\kappa) \equiv (\kappa a d)^2 \, \mathcal{I}(d;\nu,\kappa) - c^2 - (\nu c + \kappa b d)^2 \, \mathcal{K}(d;\nu,\kappa) ,$$

$$\mathcal{I}_{\nu\phi}(r) \equiv \mathcal{I}(r;\nu m,\nu k) , \quad \mathcal{K}_{\nu\phi}(r) \equiv \mathcal{K}(r;\nu m,\nu k) , \quad \nu = 1,2,\ldots .$$

In expressions which are clear from the context involving no coordinate variables, $\mathcal{I}_\phi$, $\mathcal{I}_{2\phi}$, $\mathcal{K}_\phi$, $\mathcal{K}_{2\phi}$ will be understood as evaluated at $r = d$. Also we shall use equal signs to mean "equal to the real part of" whenever the right side is complex-valued, and use asterisks to mean complex conjugates.

The plasma-vacuum interface is a free surface with helical structure. When the helical corrugation is small $\rho(\phi)$ can be written

$$\rho = d\left[1 + \varepsilon \exp(i\phi) + \frac{1}{2}\varepsilon^2\lambda \exp(i2\phi) + O(\varepsilon^3)\right] .$$

The small parameter ε is a measure of the helical corrugation, and the constant λ is to be determined by the boundary conditions. In the plasma region $(0 < r < \rho)$ we take $\Lambda = ma$ and write a solution of Eq. (1) as

$$\Psi = a_0 + ka\frac{r^2}{2} + \varepsilon a_1 r \frac{d}{dr} \mathcal{I}_\phi \exp(i\phi) + \frac{1}{4}\varepsilon^2 a_2 r \frac{d}{dr} \mathcal{I}_{2\phi} \exp(i2\phi) + O(\varepsilon^3) .$$

The corresponding magnetic field is

$$\vec{B} = (0,0,a) + \varepsilon a_1\left(-i \frac{d}{dr} I_\phi , \frac{m}{r} I_\phi , k I_\phi\right) \exp(i\phi) \tag{7}$$
$$+ \frac{1}{2} \varepsilon^2 a_2\left(-i \frac{d}{dr} I_{2\phi} , \frac{2m}{r} I_2 , 2k I_{2\phi}\right) \exp(i2\phi) + O(\varepsilon^3) .$$

In the vacuum region ($\rho < r < e$) we take $\Lambda = mb - kcd$ and write another solution of Eq. (1) as

$$\Psi = b_0 + mcd \log \frac{r}{e} + kb \frac{r^2-e^2}{2} + \varepsilon b_1 r \frac{d}{dr} K_\phi \exp(i\phi)$$
$$+ \frac{1}{4} \varepsilon^2 b_2 r \frac{d}{dr} K_{2\phi} \exp(i2\phi) + O(\varepsilon^3) .$$

The corresponding magnetic field is

$$\vec{B} = (0,c \frac{d}{r} ,b) + \varepsilon b_1\left(-i \frac{d}{dr} K_\phi , \frac{m}{r} K_\phi \; k K_\phi\right) \exp(i\phi) \tag{8}$$
$$+ \frac{1}{2} \varepsilon^2 b_2\left(-i \frac{d}{dr} K_{2\phi} , \frac{2m}{r} K_{2\phi} , 2k K_{2\phi}\right) \exp(i2\phi) + O(\varepsilon^3) .$$

The various coefficients will be determined in terms of the four physical parameters A, B, C, and D by the boundary conditions.

The constants a_0 and b_0 are related to the magnetic fluxes by

$$a_0 = - \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\rho} (mB_\theta + krB_z)\, dr ,$$

$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_\rho^{e} (mB_\theta + krB_z)\, dr .$$

From the requirement that Ψ is constant at the interface we find

$$a_1 = -\,kad^2 + O(\varepsilon^2)\ ,$$

$$b_1 = -\,(mc+kbd)d + O(\varepsilon^2)\ ,$$

$$a_2 = -\,kad^2\left[1 - 2(m^2+k^2d^2)\,I_\phi + 2\lambda\right] + O(\varepsilon)\ ,$$

$$b_2 = 2mcd - (mc+kbd)d\left[1 - 2(m^2+k^2d^2)K_\phi + 2\lambda\right] + O(\varepsilon)\ .$$

Next, from the requirement that $\left[\frac{1}{2}\,\vec{B}\cdot\vec{B}\right]_{r=\rho-0}^{r=\rho+0}$ is a constant, equal to the plasma pressure p, we find

(9) $W(m,k) = 0\ ,$

$$\lambda = \left[W(2m,2k)\right]^{-1}\Big\{-(kad)^2\left[\frac{3}{2} + 2\,I_{2\phi} - (m^2+k^2d^2)\,I_\phi(\tfrac{1}{2}\,I_\phi + 4\,I_{2\phi})\right]$$

$$-\,\frac{3}{2}\,c^2 - 2mc(mc+kbd)(K_\phi + 2K_{2\phi})$$

$$+\,(mc+kbd)^2\left[\frac{3}{2} + 2K_{2\phi} - (m^2+k^2d^2)\,K_\phi(\tfrac{1}{2}\,K_\phi + 4K_{2\phi})\right]\Big\}\ ,$$

and the plasma pressure is given by

(10) $$p = \frac{1}{2}\,(-a^2+b^2+c^2) + \frac{1}{4}\,\varepsilon^2\Big\{-k^2a^2d^2\left[-1 + (m^2+k^2d^2)\,I_\phi^2\right]$$

$$+\,3c^2 + 4mc(mc+kbd)K_\phi + (mc+kbd)^2\left[-1+(m^2+k^2d^2)K_\phi^2\right]\Big\} + O(\varepsilon^3)\,.$$

Finally, a, b, c, and d are related to A, B, C, and D by

(11) $$A = a\pi d^2 + \varepsilon^2 a\pi d^2(\tfrac{1}{2} - k^2d^2\,I_\phi) + O(\varepsilon^3)\ ,$$

(12) $$B = b\pi(e^2-d^2) + \varepsilon^2\pi d^2\left[-\,\tfrac{1}{2}\,b+(mc+bkd)kd\,K_\phi\right] + O(\varepsilon^3)\ ,$$

(13) $$C = Lcd\,\log\frac{e}{d} + \varepsilon^2 Ld\left[\tfrac{1}{4}\,c - \tfrac{1}{2}\,mkad\,I_\phi + \tfrac{1}{2}\,m(mc+bkd)K_\phi\right] + O(\varepsilon^3)\,,$$

(14) $$D = L\pi d^2\left[1 + \tfrac{1}{2}\,\varepsilon^2 + O(\varepsilon^3)\right]\ .$$

We have calculated the equilibrium up to second order in ε, as the stability is determined in that order.

III. Bifurcation points

Now suppose that the magnetic fluxes and the amount of plasma, (viz., A, B, C, and D) are prescribed in addition to the size of the cylindrical wall (viz., e and L). Circular configuration is not the only possible equilibrium. Helical configuration is another possibility. If the configuration is helical, the equilibrium is as described in Section II. The parameters a, b, c, d, and ε are determined in terms of A, B, C, D, e, and L by Eqs. (9), (11), (12), (13) and (14). If the configuration is circular, the plasma column has a circular cross section $\rho = d'$, the magnetic field is $\vec{B} = (0,0,a')$ in the plasma region and $\vec{B} = (0, c'd'/r', b')$ in the vacuum region. The parameters a', b', c', and d' are determined in terms of A, B, C, D, e, and L by $a'\pi d'^2 = A$, $b'\pi(e^2-d'^2) = B$. $Lc'd'\log(e/d') = C$, and $L\pi d'^2 = D$. Thus the two sets of parameters are related by

$$(15)\quad a' = a\left[1 - \varepsilon^2 k^2 d^2\, I_\phi + O(\varepsilon^3)\right],$$

$$(16)\quad b' = b + \varepsilon^2 (mc+kbd)k\, d^3 (e^2-d^2)^{-1} K_\phi + O(\varepsilon^3),$$

$$(17)\quad c' = c + \varepsilon^2\left\{-\frac{1}{4}c + \frac{1}{2}\left[-mkad\, I_\phi + c + m(mc+kbd)K_\phi\right]\left(\log\frac{e}{d}\right)^{-1}\right\} + O(\varepsilon^3)$$

$$(18)\quad d' = d\left[1 + \frac{1}{4}\varepsilon^2 + O(\varepsilon^3)\right].$$

When the circular equilibrium is subjected to a disturbance in the mode with wavenumbers m and k, its stability is determined by the algebraic sign of

$$(19)\quad (ka'd')^2\, I(d';m,k) - c'^2 - (mc'+kb'd')^2 K(d';m,k).$$

At the bifurcation point ($\varepsilon = 0$), a, b, c, and d are equal to a', b', c', and d' respectively, and expression (19) becomes $W(m,k)$ which is equal to zero according to Eq. (9). Thus, bifurcation is possible only when the values of A, B, C, and D are such that the circular equilibrium is neutrally stable for some mode of disturbance. Since near the bifurcation point the bifurcated equilibrium differs from the bifurcating equilibrium only slightly, the bifurcated equilibrium is likely unstable against the very disturbances which render the original bifurcating equilibrium unstable. Therefore, the bifurcated equilibrium can possibly be stable only when the bifurcation point is in the neighborhood of the margin of stability for the bifurcating equilibrium. The circular equilibrium is at the margin of stability when $W(m,k)$ as a function of m and k attains a minimum equal to zero. It is known[6,7] that allowing k to be any value (ignoring the requirement that k must be a multiple of $2\pi/L$), the minimum of $W(m,k)$ occurs at $m = 1$ or $m = 0$. Furthermore the minimum of $W(0,k)$ occurs at $k = 0$, and its minimum value is $2a^2-c^2+2b^2d^2 / (e^2-d^2)$. The helical structure disappears completely when $m = 0$ and $k = 0$ at the same time. Therefore, a stable helical equilibrium requires

(20) $$m = 1 ,$$

and the parameters must be such that

(21) $$2a^2 - c^2 + 2b^2 \frac{d^2}{e^2-d^2} > 0 .$$

Also the plasma pressure has to be positive. This requires

(22) $$- a^2 + b^2 + c^2 > 0 .$$

Let us consider a parameter space for the three ratios A/C, B/C, and $L\pi e^2/D$ (or equivalently in a parameter space for the three ratios a^2/c^2, b^2/c^2, and e/d, see Fig. 1 and Appendix A). The simultaneous conditions of $W = 0$ and $\partial W/\partial k = 0$ (upon elimination of k, with m = 1) define a surface in this parameter space. A circular equilibrium represented by a point in this surface is marginally stable. Circular equilibria represented by points in the region above the surface of marginal stability are unstable. Any point in this region of instability for circular equilibria can be a point of bifurcation. However, only a set of discrete points will have the associated value of k equal to some multiple of $2\pi/L$. As ε varies among a class of helical equilibria with the same value of m and the same value of k, a curve is mapped out in the parameter space. The point with $\varepsilon = 0$ is the bifurcation point. Such a curve may stay in the region of instability or may penetrate the surface of marginal stability and enter into the region of stability. This depends on the stability of the corresponding circular equilibrium which is determined by the algebraic sign of

$$(a'-a)\,\frac{\partial W}{\partial a} + (b'-b)\,\frac{\partial W}{\partial b} + (c'-c)\,\frac{\partial W}{\partial c} + (d'-d)\,\frac{\partial W}{\partial d} \,.$$

This is the Taylor expansion of (19) at the bifurcation point.

As was mentioned previously, a stable helical equilibrium must have its bifurcation point in the neighborhood of the surface of marginal stability in the parameter space. Moreover, any instability for such a bifurcated helical equilibrium must arise from disturbances with an azimuthal wavenumber m' equal to m, viz., 1 and an axial wavenumber k' differing from k at most by an amount first order in ε.

IV. Stability Analysis

According to the energy principle for stability, an equilibrium is stable if all disturbances will increase its potential energy. For a hydromagnetic configuration with a sharp plasma-vacuum interface, the second variation of the potential energy[3,8] is given by

$$\delta W = W_P + W_S + W_V \ ,$$

where

$$W_P = \iiint_{0<r<\rho} \frac{1}{2}\left[\Gamma p(\nabla\cdot\vec{\xi})^2 + \vec{Q}\cdot\vec{Q}\right] d\tau \ ,$$

$$W_S = \iint_{r=\rho} \frac{1}{2}(\vec{n}\cdot\vec{\xi})\left[\vec{n}\cdot\nabla \ \frac{1}{2} \ \vec{B}\cdot\vec{B}\right]_P^V \vec{\xi}\cdot d\vec{S} \ ,$$

$$W_V = \iint_{\rho<r<c} \frac{1}{2} \ \nabla\Phi \cdot \nabla\Phi \ d\tau \ .$$

$\vec{\xi}$ is the displacement of the plasma due to the disturbance, Γ is the ratio of specific heats, and

$$\vec{Q} \equiv \nabla \times (\vec{\xi}\times\vec{B}) \ .$$

$\vec{n}$ is a unit normal vector at the interface. $\nabla\Phi$ is the additional magnetic field in the vacuum region due to the disturbance.

Let

$$\vec{N} \equiv \left(\rho, \ -\frac{\partial\rho}{\partial\theta} \ , \ -\rho \ \frac{\partial\rho}{\partial z}\right) \ ,$$

which is an outward vector normal to the interface so that $d\vec{S} = \vec{N} \ d\theta \ dz$. Hence,

$$W_S = \int_0^L \int_0^{2\pi} \frac{1}{2} (\vec{N}\cdot\vec{\xi})^2_{r=\rho} \left[\frac{\vec{N}}{\vec{N}\cdot\vec{N}} \cdot \nabla \frac{\vec{B}\cdot\vec{B}}{2} \right]_{r=\rho-0}^{r=\rho+0} d\theta \, dz \quad (23)$$

When $\vec{\xi}$ satisfies the Euler-Lagrange equation:

$$\Gamma p \nabla(\nabla\cdot\vec{\xi}) - \vec{B} \times \nabla \times \nabla \times (\vec{\xi}\times\vec{B}) = 0 , \quad (24)$$

W_P is minimized to

$$W_P = \int_0^L \int_0^{2\pi} \frac{1}{2} (\Gamma p \nabla\cdot\vec{\xi} - \vec{B}\cdot\vec{Q})_{r=\rho} (\vec{N}\cdot\vec{\xi})_{r=\rho} d\theta \, dz . \quad (25)$$

The magnetic field $\nabla\Phi$ must satisfy the boundary conditions

$$\frac{\partial}{\partial r} \Phi \Big|_{r=e} = 0 , \quad (26)$$

$$\vec{N}\cdot\nabla\Phi \Big|_{r=\rho} = \vec{N} \cdot \nabla \times (\vec{\xi}\times\vec{B}) \Big|_{r=\rho} . \quad (27)$$

Moreover, since $\nabla\Phi$ is a harmonic vector in the vacuum region, Φ satisfies the Laplace equation

$$\nabla^2\Phi = 0 . \quad (28)$$

Hence W_V can be written

$$W_V = \int_0^L \int_0^{2\pi} \frac{1}{2}(-\Phi)_{r=\rho} (\vec{N}\cdot\nabla\Phi)_{r=\rho} d\theta \, dz . \quad (29)$$

In fact W_V is minimized when Φ satisfies Eq. (28).

As was discussed in Section III we only have to scrutinize the disturbances with wavenumbers m' and k' such that

$$m' = 1 , \quad k' = k(1+\varepsilon s) . \quad (30)$$

By virtue of the bifurcation condition, the minimized δW will be second order in ε. The two sets of wavenumbers, one for the disturbance and the other for the helical equilibrium, interact and generate mixed wavenumbers appearing as $\nu\phi + \nu'\phi'$, where

$$\phi' \equiv m'\theta + k'z \ ,$$

for $\vec{\xi}$ satisfying Eq. (24). The calculations appear differently for $s = 0$ and for $s \neq 0$. In the case $k' = k$ the displacement which minimizes δW is compressible, whereas in the case $k' \neq k$ (more precisely when s is zeroth order in ε) the required displacement is incompressible. We shall consider the first case in this section and the second case in the next section. It is unnecessary to scrutinize the gap for s smaller than zeroth order in ε, if the ratio d/L is first order in ε. Because both k and k' must be a multiple of $2\pi/L$, hence their difference must be zero or at least equal to $2\pi/L$.

To simplify notations we shall put $d = 1$. In other words, k stands for kd and e stands for e/d henceforth.

First, we shall find $\vec{\xi}$ which satisfies Eq. (24). Since in the plasma region the magnetic field is given by

$$\vec{B} = \vec{B}_0 + \varepsilon\,\vec{B}_1 + \varepsilon^2\vec{B}_2 + O(\varepsilon^3)$$

with

$$\vec{B}_0 = \left(0, 0, a\right),$$

$$\vec{B}_1 = a_1\left(-iR, \Theta, Z\right)\exp(i\phi) \ , \quad R = \frac{d}{dr}\, I_\phi \ , \quad \Theta = \frac{m}{r}\, I_\phi \ , \ Z = k\, I_\phi ,$$

$$\vec{B}_2 = \frac{1}{2}\, a_2\left(-i\,\frac{d}{dr}\, I_{2\phi} \ , \ \frac{2m}{r}\, I_{2\phi} \ , \ 2k\, I_{2\phi}\right) \exp\,(i2\phi) \ ,$$

we shall expand $\vec{\xi}$ in ε formally as

$$\vec{\xi} = \vec{\xi}_0 + \varepsilon\,\vec{\xi}_1 + \varepsilon^2\vec{\xi}_2 + O(\varepsilon^3) \ .$$

By separation of variables, $\vec{\xi}_0$, $\vec{\xi}_1$ and $\vec{\xi}_2$ can be resolved into sums of terms whose dependence on θ and z has the form $\exp\,(i\nu\theta + i\kappa z)$. Only certain terms of such form are necessary to be included in the process of minimizing δW. It is convenient to define a linear partial differential operator L (see Appendix B) as

(31) $$L\,\vec{x} \equiv \Gamma p\,\nabla(\nabla\cdot\vec{x}) - \vec{B}_0 \times \nabla \times \nabla \times (\vec{x}\times\vec{B}_0)$$

Then $\vec{\xi}_0$ must satisfy

(32) $$L\,\vec{\xi}_0 = O(\varepsilon) \ .$$

As is shown in Appendix B, various harmonic vectors are orthogonal in contributions to δW. Therefore in the case $k = k'$, which is exclusively discussed in the rest of this section, it is only necessary to consider the particular solution

(33) $$\vec{\xi}_0 = \alpha\left(R^{\alpha}_{0,1}\ ,\ i\Theta^{\alpha}_{0,1}\ ,\ iZ^{\alpha}_{0,1}\right)\,\exp\,(i\phi)\ ,$$

where

$$R^{\alpha}_{0,1} = \frac{d}{dr}\ I_{\phi}\ ,\quad \Theta^{\alpha}_{0,1} = \frac{m}{r}\ I_{\phi}\ ,\quad Z^{\alpha}_{0,1} = k\ I_{\phi}\ ,$$

α being a complex-valued constant, allowing a phase difference between $\vec{B}_1$ and $\vec{\xi}_0$. Since $\nabla\cdot\vec{\xi}_0 = 0$ and $\nabla\times\nabla\times(\vec{\xi}_0\times\vec{B}_0) = 0$, $\vec{\xi}_1$ must satisfy the following equation.

(34) $$L\,\vec{\xi}_1 = \vec{B}_0 \times \nabla \times \nabla \times (\vec{\xi}_0 \times \vec{B}_1) + O(\varepsilon)\ .$$

The right side is independent of θ and z; hence so is the inhomogeneous solution, the term with the coefficient α in (35). Observe that among various terms in $\vec{\xi}_2$ only those containing exp $(i\phi)$ will interact with $\vec{\xi}_0$ which contains exp $(i\phi)$ only and contribute to δW in second order of ε. Thus $\vec{\xi}_1$ must include terms containing exp $(i2\phi)$ or exp $(i0)$. Indeed it is only necessary to consider the following particular solution.

(35) $$\vec{\xi}_1 = \frac{\alpha}{2}\,(R^{\alpha}_{1,0}\ ,0,0) + \beta(R^{\beta}_{1,0}\ ,0,0)$$
$$+\ \gamma(R^{\gamma}_{1,2}\ ,\ i\Theta^{\gamma}_{1,2}\ ,\ iZ^{\gamma}_{1,2})\ \exp\ (i2\phi)\ ,$$

where

$$R^{\alpha}_{1,0} = \frac{a^2k}{\Gamma p+a^2}\,(R^{\alpha}_{0,1}Z + Z^{\alpha}_{0,1}R)\ ,$$

$$R^{\beta}_{1,0} = \frac{r}{2}\ ,$$

$$R^{\gamma}_{1,2} = \frac{d}{dr}\ I_{2\phi}\ ,\quad \Theta^{\gamma}_{1,2} = \frac{2m}{r}\ I_{2\phi}\ ,\quad Z^{\gamma}_{1,2} = 2k\ I_{2\phi}\ ,$$

β being a real-valued constant and γ being a complex-valued constant. Other harmonic vectors, if included in $\vec{\xi}_1$, will make a positive contribution to δW in second order of ε, hence may be excluded in minimizing δW. Now $\vec{\xi}_2$ must satisfy the following equation.

(36) $$L\,\vec{\xi}_2 = \vec{B}_0\times\nabla\times\nabla\times(\vec{\xi}_0\times\vec{B}_2+\vec{\xi}_1\times\vec{B}_1)$$
$$+\ \vec{B}_1\times\nabla\times\nabla\times(\vec{\xi}_0\times\vec{B}_1+\vec{\xi}_1\times\vec{B}_0) + O(\varepsilon)\ .$$

A particular solution of

$$L\,\vec{x} = \vec{B}_1 \times \nabla \times \nabla \times (\vec{\xi}_0 \times \vec{B}_1 + \vec{\xi}_1 \times \vec{B}_0)$$

is

$$\vec{x} = \frac{\alpha}{4}\,(X_r\,,-iX_\theta\,,-iX_z)\ \exp\,(-i\phi) + \frac{\alpha}{4}\,(X_r\,,iX_\theta\,,iX_z)\ \exp\,(i\phi)\ ,$$

provided that X_r , X_θ and X_z satisfy

$$a^2k(\frac{d}{dr}\,X_z - kX_r) + (\Gamma p + a^2)\,\frac{d}{dr}\,(\frac{1}{r}\,\frac{d}{dr}\,r\,X_r - \frac{m}{r}\,X_\theta - kX_z)$$

$$= -a^2k^2(\Theta S_\theta + \frac{\Gamma p}{\Gamma p + a^2}\,Z\,S_z)\ ,$$

$$a^2k(\frac{m}{r}\,X_z - kX_\theta) + (\Gamma p + a^2)\,\frac{m}{r}\,(\frac{1}{r}\,\frac{d}{dr}\,r\,X_r - \frac{m}{r}\,X_\theta - kX_z)$$

$$= -\,a^2k^2R\,S_\theta\ ,$$

$$(\Gamma p + a^2)\,k(\frac{1}{r}\,\frac{d}{dr}\,r\,X_r - \frac{m}{r}\,X_\theta - kX_z) = -\,a^2k^2R\,S_z\ ,$$

where

$$S_\theta = \frac{1}{r}\,\frac{d}{dr}\,r\left[\frac{d}{dr}(R^\alpha_{0,1}\Theta + \Theta^\alpha_{0,1}R)\right]\ , \qquad S_z = \frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\,r(R^\alpha_{0,1}Z + Z^\alpha_{0,1}R)\right]\ .$$

Hence

$$X_r = \Theta\,S_\theta + \frac{\Gamma p}{\Gamma p + a^2}\,Z\,S_z + \frac{d}{dr}\,S_r\ ,$$

$$X_\theta = R\,S_\theta + \frac{m}{r}\,S_r\ ,$$

$$X_z = R\,S_z + k\,S_r\ ,$$

where

$$S_r = \int_0^r \frac{1}{k} \frac{I_m(kr)K_m(k\tau) - K_m(kr)I_m(k\tau)}{I_m'(k\tau)K_m(k\tau) - K_m'(k\tau)I_m(k\tau)} T(\tau)\, d\tau$$

is a regular solution of

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2} - k^2\right) S_r = T(r) \equiv -\Theta \frac{d}{dr} S_\theta - \frac{\Gamma p}{\Gamma p + a^2} Z \frac{1}{r}\frac{d}{dr} r S_z \ .$$

On the other hand, a particular solution of

$$L\ \vec{y} = \vec{B}_0 \times \nabla \times \nabla \times (\vec{\xi}_0 \times \vec{B}_2 + \vec{\xi}_1 \times \vec{B}_1)$$

is

$$\vec{y} = \frac{\alpha}{4}\left(Y_r + Z_r\ ,\ -iY_\theta - iZ_\theta\ ,\ -iY_z - iZ_z\right) \exp(-i\phi)$$

$$+\left[\frac{\alpha}{4}(Y_r, iY_\theta, iY_z) + \beta(R^\beta_{2,1}, i\Theta^\beta_{2,1}, iZ^\beta_{2,1}) + \frac{\gamma}{2}(R^\gamma_{2,1}, i\Theta^\gamma_{2,1}, iZ^\gamma_{2,1})\right] \exp(i\phi)$$

$$+\left[\frac{\alpha}{4}(R^\alpha_{2,3}, i\Theta^\alpha_{2,3}, iZ^\alpha_{2,3}) + \frac{\gamma}{2}(R^\gamma_{2,3}, i\Theta^\gamma_{2,3}, iZ^\gamma_{2,3})\right] \exp(i3\phi)\ ,$$

provided that $\vec{y}$ satisfies

$$-\ a \frac{\partial}{\partial x} \vec{y} = \nabla \times (\vec{\xi}_0 \times \vec{B}_2 + \vec{\xi}_1 \times \vec{B}_1)\ .$$

Hence

$$Y_r = R^\alpha_{1,0}\left(\frac{m}{r}\Theta + kZ\right)\ ,$$

$$Y_\theta = \frac{d}{dr} R^\alpha_{1,0}\Theta\ ,$$

$$Y_z = \frac{1}{r}\frac{d}{dr} r\, R^\alpha_{1,0} Z\ ,$$

$$Z_r = -\frac{a_2}{ak}\left[R^\alpha_{0,1}\left(\frac{2m^2}{r^2} + 2k^2\right) I_{2\phi} + \left(\frac{m}{r}\Theta^\alpha_{0,1} + kZ^\alpha_{0,1}\right)\frac{d}{dr} I_{2\phi}\right]\ ,$$

$$Z_\theta = -\frac{a_2}{ak}\frac{d}{dr}\left(R^\alpha_{0,1}\frac{2m}{r} I_{2\phi} + \Theta^\alpha_{0,1}\frac{d}{dr} I_{2\phi}\right)\ ,$$

$$Z_z = -\frac{a_2}{ak}\frac{1}{r}\frac{d}{dr} r\left(R^\alpha_{0,1}\, 2k\, I_{2\phi} + Z^\alpha_{0,1}\frac{d}{dr} I_{2\phi}\right)\ ,$$

where

$$R^{\beta}_{2,1} = R^{\beta}_{1,0}(\frac{m}{r}\,\Theta + kZ) \ ,$$

$$\Theta^{\beta}_{2,1} = \frac{d}{dr}\, R^{\beta}_{1,0}\Theta \quad ,$$

$$Z^{\beta}_{2,1} = \frac{1}{r}\,\frac{d}{dr}\, r\, R^{\beta}_{1,0}Z \quad ,$$

$$R^{\gamma}_{2,1} = R^{\gamma}_{1,2}(\frac{m}{r}\,\Theta + kZ) + (\frac{m}{r}\,\Theta^{\gamma}_{1,2} + kZ^{\gamma}_{1,2})R \ ,$$

$$\Theta^{\gamma}_{2,1} = \frac{d}{dr}\,(R^{\gamma}_{1,2}\Theta + \Theta^{\gamma}_{1,2}R\ ) \ ,$$

$$Z^{\gamma}_{2,1} = \frac{1}{r}\,\frac{d}{dr}\, r(R^{\gamma}_{1,2}Z + Z^{\gamma}_{1,2}R) \ .$$

Any harmonic vectors, if included in $\vec{\xi}_2$ as a homogeneous part, will make contribution to δW in third order of ε. Therefore, it is only necessary to consider the following particular solution

$$(37)\quad \vec{\xi}_2 = \vec{x} + \vec{y}$$

$$= \frac{\alpha}{4}\,(R^{\alpha}_{2,-1}, i\Theta^{\alpha}_{2,-1}, iZ^{\alpha}_{2,-1})\ \exp\,(-i\phi)$$

$$+ \Big[\frac{\alpha}{4}(R^{\alpha}_{2,1}, i\Theta^{\alpha}_{2,1}, iZ^{\alpha}_{2,1}) + \beta(R^{\beta}_{2,1}, i\Theta^{\beta}_{2,1}, iZ^{\beta}_{2,1})$$

$$+ \frac{\gamma}{2}(R^{\gamma}_{2,1}, i\Theta^{\gamma}_{2,1}, iZ^{\beta}_{2,1})\Big]\ \exp\,(i\phi)$$

$$+ \Big[\frac{\alpha}{4}(R^{\alpha}_{2,3}, i\Theta^{\alpha}_{2,3}, iZ^{\alpha}_{2,3}) + \frac{\gamma}{2}(R^{\gamma}_{2,3}, i\Theta^{\gamma}_{2,3}, iZ^{\gamma}_{2,3})\Big]\exp(i3\phi) \ ,$$

where

$$R^{\alpha}_{2,-1} = X_r + Y_r + Z_r, \quad \Theta^{\alpha}_{2,-1} = -X_\theta - Y_\theta - Z_\theta, \quad Z^{\alpha}_{2,-1} = -X_z - Y_z - Z_z \ ,$$

$$R^{\alpha}_{2,1} = X_r + Y_r \ , \qquad \Theta^{\alpha}_{2,1} = X_\theta + Y_\theta \ , \quad Z^{\alpha}_{2,1} = X_z + Y_z \ .$$

With the above solution for $\vec{\xi}$ we have

$$(38)\quad \nabla\cdot\vec{\xi} = \varepsilon\left(\frac{\alpha}{2}\frac{1}{r}\frac{d}{dr}\, r\, R^{\alpha}_{1,0}+\beta\right) - \varepsilon^2\,\frac{\alpha^*+\alpha}{4}\,\frac{k}{\Gamma p+a^2} S_z R\,\exp(i\phi)+O(\varepsilon^3),$$

and

$$(39)\quad Q = \alpha ak\left(iR^{\alpha}_{0,1},-\Theta^{\alpha}_{0,1},-Z^{\alpha}_{0,1}\right)\,\exp\,(i\phi)$$

$$+\ \varepsilon\Big[\frac{\alpha}{2}\,ak\Big(0,\ \frac{d}{dr}(R^{\alpha}_{0,1}\Theta+\Theta^{\alpha}_{0,1}R),\ \frac{\Gamma p}{\Gamma p+a^2}\,\frac{1}{r}\frac{d}{dr}\, r(R^{\alpha}_{0,1}Z+Z^{\alpha}_{0,1}R)\Big)$$

$$+\ \beta\left(0,0,-a\right) + 2\gamma ak\left(iR^{\gamma}_{1,2},-\Theta^{\gamma}_{1,2},-Z^{\gamma}_{1,2}\right)\,\exp\,(i2\phi)\Big]$$

$$+\ \varepsilon^2\,\frac{\alpha^*+\alpha}{4}\,ak\left(iX_r,-X_\theta,-X_z+\frac{a^2}{\Gamma p+a^2}\,S_zR\right)\,\exp\,(i\phi) + O(\varepsilon^3)\ .$$

Next we shall find Φ which satisfies Eqs. (26), (27) and (28). In the vacuum region the magnetic field is given by Eq. (8), thus a straightforward computation gives

$$(40)\quad \vec{N}\cdot\nabla\times(\vec{\xi}\times\vec{B})\Big|_{r=\rho} = i\alpha E^{\alpha}_{0,1}\,\exp\,(i\phi) + i\varepsilon\left(\frac{\alpha}{2}\,E^{\alpha}_{1,2}+\gamma E^{\gamma}_{1,2}\right)\exp(i2\phi)$$

$$+i\varepsilon^2\Big[\frac{\alpha}{4}\,E^{\alpha}_{2,-1}\exp(-i\phi)+\left(\frac{\alpha}{4}\,E^{\alpha}_{2,1}+\beta E^{\beta}_{2,1}+\frac{\gamma}{2}\,E^{\gamma}_{2,1}\right)\exp(i\phi)$$

$$+\ \left(\frac{\alpha}{4}\,E^{\alpha}_{2,3}+\frac{\gamma}{2}\,E^{\gamma}_{2,3}\right)\exp(i3\phi)\Big] + O(\varepsilon^3)\ ,$$

The various coefficients, E's in (40) as well as F's in (41), A's in (43), B's in (44), and D's in (46), are given in Appendix C. In view of (40), Φ satisfying Eqs. (26), (27) and (28) is

$$(41)\quad \Phi = i\alpha F^{\alpha}_{0,1} K_\phi \exp(i\phi) + i\varepsilon\left(\frac{\alpha}{2} F^{\alpha}_{1,2} + \gamma F^{\gamma}_{1,2}\right) K_{2\phi} \exp(i2\phi)$$

$$+ i\varepsilon^2 \left[\frac{\alpha}{4} F^{\alpha}_{2,-1} K_\phi \exp(-i\phi) + \left(\frac{\alpha}{4} F^{\alpha}_{2,1} + \beta F^{\beta}_{2,1} + \frac{\gamma}{2} F^{\gamma}_{2,1}\right) K_\phi \exp(i\phi)\right.$$

$$\left. + \left(\frac{\alpha}{4} F^{\alpha}_{2,3} + \frac{\gamma}{2} F^{\gamma}_{2,3}\right) K_{3\phi} \exp(i3\phi)\right] + O(\varepsilon^3) \ .$$

Now in order to evaluate the integrals (23), (25), and (29), the following are computed.

$$(42)\quad \left[\frac{\vec{N}}{\vec{N}\cdot\vec{N}} \cdot \nabla \frac{\vec{B}\cdot\vec{B}}{2}\right]_{r=\rho-0} = C_0 + \frac{1}{2}\varepsilon C_1 \exp(i\phi)$$

$$+ \varepsilon^2 \left[\frac{1}{4} C_2 + \frac{1}{8} C_3 \exp(i2\phi)\right] + O(\varepsilon^3) \ ,$$

where

$$C_0 = -c^2 \ ,$$

$$C_1 = 2a^2k^2 + 8c^2 + 4mc(mc+kb)K_\phi - 2(mc+kb)^2 \ ,$$

$$C_2 = -2a^2k^2\left[1 + 2(m^2+k^2) I_\phi - m^2 I_\phi^2\right] + 2c^2(-10+m^2+k^2)$$

$$+ 8mc(mc+kb)(1-2K_\phi) + 2(mc+kb)^2\left[1+2(m^2+k^2)K_\phi - m^2K_\phi^2\right] \ ,$$

$$C_3 = -a^2k^2\left[1+2(m^2+k^2) I_\phi - m^2 I_\phi^2\right] - c^2(10+8m^2K_{2\phi}+m^2+k^2)$$

$$+ 4mc(mc+kb)\left[2-2K_\phi+K_{2\phi}-2(m^2+k^2)K_\phi K_{2\phi}\right]$$

$$+ (mc+kb)^2\left[1+2(m^2+k^2)K_\phi - m^2K_\phi^2\right]$$

$$+ 4\lambda\left[a^2k^2+c^2+2mc(mc+kb)K_{2\phi} - (mc+kb)^2\right] \ .$$

$$(43)\quad (\vec{N}\cdot\vec{\xi})_{r=\rho} = \alpha A^{\alpha}_{0,1}\exp(i\phi) + \varepsilon\left[\frac{\alpha}{2} A^{\alpha}_{1,0} + \beta A^{\beta}_{1,0} + \left(\frac{\alpha}{2} A^{\alpha}_{1,2} + \gamma A^{\gamma}_{1,2}\right)\exp(i2\phi)\right]$$

$$+\varepsilon^2\left[\frac{\alpha}{4} A^{\alpha}_{2,-1}\exp(-i\phi) + \left(\frac{\alpha}{4} A^{\alpha}_{2,1} + \beta A^{\beta}_{2,1} + \frac{\gamma}{2} A^{\gamma}_{2,1}\right)\exp(i\phi)\right.$$

$$\left. + \left(\frac{\alpha}{4} A^{\alpha}_{2,3} + \frac{\gamma}{2} A^{\gamma}_{2,3}\right)\exp(i3\phi)\right] + O(\varepsilon^3) \ ,$$

$$(44) \quad (\Gamma p\ \nabla\cdot\vec{\xi} - \vec{B}\cdot\vec{Q})_{r=\rho} = \alpha\ B^{\alpha}_{0,1}\exp(i\phi)$$

$$+ \varepsilon\left[\frac{\alpha}{2} B^{\alpha}_{1,0} + \beta B^{\beta}_{1,0} + \left(\frac{\alpha}{2} B^{\alpha}_{1,2} + \gamma B^{\gamma}_{1,2}\right) \exp\ (i2\phi)\right]$$

$$+ \varepsilon^2\left[\frac{\alpha}{4} B^{\alpha}_{2,-1}\exp(-i\phi) + \left(\frac{\alpha}{4} B^{\alpha}_{2,1} + \beta B^{\beta}_{2,1} + \frac{\gamma}{2} B^{\gamma}_{2,1}\right)\exp(i\phi)\right.$$

$$\left. + \left(\frac{\alpha}{4} B^{\alpha}_{2,3} + \frac{\gamma}{2} B^{\gamma}_{2,3}\right)\exp(i3\phi)\right] + O(\varepsilon^3) \ ,$$

$$(45) \quad (\vec{N}\cdot\vec{\xi})^2_{r=\rho} = \frac{1}{2}\alpha^*\alpha A^{\alpha}_{0,1}A^{\alpha}_{0,1} + \frac{1}{2}\alpha\alpha A^{\alpha}_{0,1}A^{\alpha}_{0,1}\exp(i2\phi)$$

$$+\varepsilon\left\{\left[\frac{1}{2}\alpha^*\alpha A^{\alpha}_{0,1}(A^{\alpha}_{1,0}+A^{\gamma}_{1,2}) + \frac{1}{2}\alpha\alpha A^{\alpha}_{0,1}A^{\alpha}_{1,0}\right.\right.$$

$$\left. + 2\alpha\beta A^{\alpha}_{0,1}A^{\beta}_{1,0} + \alpha^*\gamma A^{\alpha}_{0,1}A^{\gamma}_{1,2}\right]\exp(i\phi)$$

$$\left. + \left(\frac{1}{2}\alpha\alpha A^{\alpha}_{0,1}A^{\alpha}_{1,2} + \alpha\gamma A^{\alpha}_{0,1}A^{\gamma}_{1,2}\right) \exp\ (i3\phi)\right\}$$

$$+ \varepsilon^2\left[\frac{1}{4}\alpha^*\alpha\left(A^{\alpha}_{0,1}A^{\alpha}_{2,1} + \frac{1}{2}A^{\alpha}_{1,0}A^{\alpha}_{1,0} + \frac{1}{2}A^{\alpha}_{1,2}A^{\alpha}_{1,2}\right)\right.$$

$$+ \frac{1}{4}\alpha\alpha\left(A^{\alpha}_{0,1}A^{\alpha}_{2,-1} + \frac{1}{2}A^{\alpha}_{1,0}A^{\alpha}_{1,0}\right)$$

$$+ \alpha^*\beta\left(A^{\alpha}_{0,1}A^{\beta}_{2,1} + A^{\alpha}_{1,0}A^{\beta}_{1,0}\right) + \frac{1}{2}\alpha^*\gamma\left(A^{\alpha}_{0,1}A^{\gamma}_{2,1} + A^{\alpha}_{1,2}A^{\gamma}_{1,2}\right)$$

$$+ \beta\beta\ A^{\beta}_{1,0}A^{\beta}_{1,0} + \frac{1}{2}\gamma^*\gamma A^{\gamma}_{1,2}A^{\gamma}_{1,2}$$

$$\left. + \text{const. } \exp\ (i2\phi) + \text{const. } \exp\ (i4\phi)\right] + O(\varepsilon^3) \ ,$$

$$(46)\quad (-\Phi)_{r=\rho} = i\alpha D^{\alpha}_{0,1}\exp(i\phi) + i\varepsilon\left[\frac{\alpha}{2} D^{\alpha}_{1,0}+\left(\frac{\alpha}{2} D^{\alpha}_{1,2}+\gamma D^{\gamma}_{1,2}\right)\exp(i2\phi)\right]$$

$$+ i\varepsilon^2\left[\frac{\alpha}{4} D^{\alpha}_{2,-1}\exp(-i\phi) + \left(\frac{\alpha}{4} D^{\alpha}_{2,1} + \beta D^{\beta}_{2,1} + \frac{\gamma}{2} D^{\gamma}_{2,1}\right) \exp(i\phi)\right.$$

$$\left. + \left(\frac{\alpha}{4} D^{\alpha}_{2,3} + \frac{\gamma}{2} D^{\gamma}_{2,3}\right) \exp(i3\phi)\right] + O(\varepsilon^3) \ ,$$

and $(\vec{N}\cdot\nabla\Phi)_{r=\rho}$ is given by (40).

Only the constant terms independent of θ and z in the integrands contribute to the integrals (23), (25) and (29). Thus we obtain

$$(47)\quad \delta W = \frac{\pi L}{2}\, \alpha^*\alpha\ W(m,k)$$

$$+ \varepsilon^2\ \frac{\pi L}{2}\left[\frac{1}{4}\ \alpha^*\alpha W^{\alpha\alpha} + \frac{1}{4}\ \alpha\alpha W'^{\alpha\alpha} + \alpha\beta W^{\alpha\beta} + \frac{1}{2}\ \alpha^*\gamma W^{\alpha\gamma}\right.$$

$$\left. + 2\beta\beta\ W^{\beta\beta} + \gamma^*\gamma\ W^{\gamma\gamma}\right] + O(\varepsilon^3) \ ,$$

where

$$W^{\alpha\alpha} = A^{\alpha}_{0,1}B^{\alpha}_{2,1} + A^{\alpha}_{1,0}B^{\alpha}_{1,0} + A^{\alpha}_{1,2}B^{\alpha}_{1,2} + A^{\alpha}_{2,1}B^{\alpha}_{2,1}$$

$$+ C_0(2A^{\alpha}_{0,1}A^{\alpha}_{2,1} + A^{\alpha}_{1,0}A^{\alpha}_{1,0} + A^{\alpha}_{1,2}A^{\alpha}_{1,2}) + C_1A^{\alpha}_{0,1}(A^{\alpha}_{1,0}+A^{\alpha}_{1,2})$$

$$+ C_2A^{\alpha}_{0,1}A^{\alpha}_{0,1} + D^{\alpha}_{0,1}E^{\alpha}_{2,1} + D^{\alpha}_{1,2}E^{\alpha}_{1,2} + D^{\alpha}_{2,1}E^{\alpha}_{0,1} \ ,$$

$$W'^{\alpha\alpha}= A^{\alpha}_{0,1}B^{\alpha}_{2,-1} + A^{\alpha}_{2,-1}B^{\alpha}_{0,1} + A^{\alpha}_{1,0}B^{\alpha}_{1,0} + C_0(2A^{\alpha}_{0,1}A^{\alpha}_{2,-1}+ A^{\alpha}_{1,0}A^{\alpha}_{1,0})$$

$$+ C_1A^{\alpha}_{0,1}A^{\alpha}_{1,0} + C_3A^{\alpha}_{0,1}A^{\alpha}_{0,1} - D^{\alpha}_{0,1}E^{\alpha}_{2,-1} - D^{\alpha}_{2,-1}E^{\alpha}_{0,1} \ ,$$

$$W^{\alpha\beta} = A^{\alpha}_{0,1}B^{\beta}_{2,1} + A^{\alpha}_{1,0}B^{\beta}_{1,0} + B^{\alpha}_{1,0}A^{\beta}_{1,0} + B^{\alpha}_{0,1}A^{\beta}_{2,1}$$

$$+ 2C_0(A^{\alpha}_{0,1}A^{\beta}_{2,1} + A^{\alpha}_{1,0}A^{\beta}_{1,0}) + C_1A^{\alpha}_{0,1}A^{\beta}_{1,0} + D^{\alpha}_{0,1}E^{\beta}_{2,1}+E^{\alpha}_{0,1}D^{\beta}_{2,1} \ ,$$

$$W^{\alpha\gamma} = A^{\alpha}_{0,1}B^{\gamma}_{2,1} + A^{\alpha}_{1,2}B^{\gamma}_{1,2} + B^{\alpha}_{1,2}A^{\gamma}_{1,2} + B^{\alpha}_{0,1}A^{\gamma}_{2,1}$$

$$+ 2C_0(A^{\alpha}_{0,1}A^{\gamma}_{2,1} + A^{\alpha}_{1,2}A^{\gamma}_{1,2}) + C_1A^{\alpha}_{0,1}A^{\gamma}_{1,2}$$

$$+ D^{\alpha}_{0,1}E^{\gamma}_{2,1} + D^{\alpha}_{1,2}E^{\gamma}_{1,2} + E^{\alpha}_{1,2}D^{\gamma}_{1,2} + E^{\alpha}_{0,1}D^{\gamma}_{2,1} ,$$

$$W^{\beta\beta} = A^{\beta}_{1,0}B^{\beta}_{1,0} + C_0A^{\beta}_{1,0}A^{\beta}_{1,0} = \frac{1}{2}(\Gamma p + a^2) - \frac{1}{4}c^2 ,$$

$$W^{\gamma\gamma} = A^{\gamma}_{1,2}B^{\gamma}_{1,2} + C_0A^{\gamma}_{1,2}A^{\gamma}_{1,2} + D^{\gamma}_{1,2}E^{\gamma}_{1,2} = W(2m,2k) .$$

Finally we minimize (47) by varying the relative amplitudes and phases among the three harmonic vectors in $\vec{\xi}$. Writing

$$\alpha = (\alpha^*\alpha)^{1/2}(\cos\psi + i\sin\psi), \quad \beta = (\alpha^*\alpha)^{1/2}x_\beta, \quad \gamma = \alpha(x_\gamma + iy_\gamma)$$

and utilizing Eq. (9), we have

$$\delta W = \varepsilon^2 \frac{\pi L}{2}\alpha^*\alpha\Bigg[\frac{1}{4}W^{\alpha\alpha} - \frac{1}{16}\frac{W^{\alpha\beta}W^{\alpha\beta}}{W^{\beta\beta}} - \frac{1}{16}\frac{W^{\alpha\gamma}W^{\alpha\gamma}}{W^{\gamma\gamma}} + \left(\frac{1}{4}W'^{\alpha\alpha} - \frac{1}{16}\frac{W^{\alpha\beta}W^{\alpha\beta}}{W^{\beta\beta}}\right)\cos 2\psi$$

$$+ 2W^{\beta\beta}\left(x_\beta + \frac{1}{4}\frac{W^{\alpha\beta}}{W^{\beta\beta}}\cos\psi\right)^2 + W^{\gamma\gamma}\left(x_\gamma + \frac{1}{4}\frac{W^{\alpha\gamma}}{W^{\gamma\gamma}}\right)^2 + W^{\gamma\gamma}y_\gamma^2\Bigg] + O(\varepsilon^3) .$$

With $W^{\beta\beta} > 0$ and $W^{\gamma\gamma} > 0$, the minimum of (47) is

$$\text{(48)} \quad \delta W_{min} = \varepsilon^2 \frac{\pi L}{8}\alpha^*\alpha\Bigg[W^{\alpha\alpha} - \frac{1}{4}\frac{W^{\alpha\beta}W^{\alpha\beta}}{W^{\beta\beta}} - \frac{1}{4}\frac{W^{\alpha\gamma}W^{\alpha\gamma}}{W^{\gamma\gamma}}$$

$$- \left|W'^{\alpha\alpha} - \frac{1}{4}\frac{W^{\alpha\beta}W^{\alpha\beta}}{W^{\beta\beta}}\right|\Bigg] + O(\varepsilon^3) .$$

This is the minimum for δW when the disturbance is predominantly in the mode with $m' = 1$ and $k' = k$.

V. Stability Analysis (Continued)

In the preceding section we considered the case when the predominant mode in the disturbance has the same wave-numbers as those of the helical equilibrium. In this section we shall consider the case when the axial wavenumber of the predominant mode is different slightly from that of the helical equilibrium [$m' = 1$, $k'-k = \varepsilon sk$, $s = O(\varepsilon^0) \neq 0$]. We shall use the following notations.

$$I_{\phi'}(r) \equiv I(r;m',k') \,, \qquad K_{\phi'}(r) \equiv K(r;m',k') \,,$$

$$I_{\phi'-\phi}(r) \equiv (k'-k)^2 d^2 I(r;0,k'-k) \,, \quad K_{\phi'-\phi}(r) \equiv (k'-k)^2 d^2 K(r;0,k'-k) \,,$$

$$I_{\phi'+\phi}(r) \equiv I(r;m'+m,k'+k) \,, \qquad K_{\phi'+\phi}(r) \equiv K(r;m'+m,k'+k) \,.$$

The perturbation scheme will not be straightforward as in the preceding section because the small parameter ε will appear in such expression as $\exp i(k'-k)z$. The fact that the derivative of $\exp(i\varepsilon skz)$ is one order smaller than the function itself causes some intermixing among terms of different order in ε. Thus it is necessary to consider a particular solution of Eq. (32) with the following form

$$\vec{\xi}_0 = \vec{f}_0 + \vec{h}_{0,1} \,, \tag{49}$$

where

$$\vec{h}_{0,1} = \alpha\left(R^\alpha_{0,1}, i\Theta^\alpha_{0,1}, iZ^\alpha_{0,1}\right) \exp(i\phi')$$

with

$$R^\alpha_{0,1} = \frac{d}{dr} I_{\phi'} \,, \quad \Theta^\alpha_{0,1} = \frac{m'}{r} I_{\phi'} \,, \quad Z^\alpha_{0,1} = k' I_{\phi'}$$

is a harmonic vector, and $\vec{f}_0$ is an undetermined vector constrained by $\frac{\partial}{\partial z}\vec{f}_0 = O(\varepsilon)$ and $\nabla(\nabla\cdot\vec{f}_0) = O(\varepsilon)$. Although $\vec{f}_0$ does not contribute to δW in lower than second order of ε, its inclusion in $\vec{\xi}_0$ is essential for the expansion of $\vec{\xi}$. In fact, $\vec{f}_0$ is to be so determined that the first order equation is solvable.

Since $L\vec{h}_{0,1} = 0$, $L\vec{f}_0 = O(\varepsilon)$, and $\nabla\times\nabla\times(\vec{f}_0\times\vec{B}_0) = O(\varepsilon)$, Eq. (24) becomes

$$\varepsilon L\vec{\xi}_1 = -L\vec{f}_0 + \frac{\varepsilon}{2}\vec{B}_0\times\nabla\times\nabla\times\left|(\vec{h}_{0,1}+\vec{f}_0)\times(\vec{B}_1^*+\vec{B}_1)\right| + O(\varepsilon) .$$

In the right side the term $\frac{1}{2}\varepsilon\vec{B}_0\times\nabla\times\nabla\times(\vec{h}_{0,1}\times\vec{B}_1^*)$, which contains the factor $\exp(i\varepsilon skz)$, is not in the range of the operator εL. Therefore, solvability requires that

$$-L\vec{f}_0 + \frac{\varepsilon}{2}\vec{B}_0\times\nabla\times\nabla\times(\vec{h}_{0,1}\times\vec{B}_1^*) = O(\varepsilon^2) .$$

This is achieved provided that

$$\vec{f}_0 = \vec{g}_{0,0} + \vec{h}_{0,0} ,$$

where $\vec{h}_{0,0}$ is a harmonic vector constrained by $\frac{\partial}{\partial z}\vec{h}_{0,0} = O(\varepsilon)$, and $\vec{g}_{0,0}$ satisfies

$$a\frac{\partial}{\partial z}\vec{g}_{0,0} + \frac{\varepsilon}{2}\nabla\times(\vec{h}_{0,1}\times\vec{B}_1^*) = 0 , \quad \nabla(\nabla\cdot\vec{g}_{0,0}) = 0 .$$

Hence

$$\vec{g}_{0,0} = \frac{\alpha}{2}\left(R_{0,0}^{\alpha}, i\Theta_{0,0}^{\alpha}, iZ_{0,0}^{\alpha}\right)\exp(i\phi'-i\phi) ,$$

where

$$R_{0,0}^{\alpha} = \varepsilon k(R_{0,1}^{\alpha}Z + Z_{0,1}^{\alpha}R) ,$$

$$\Theta_{0,0}^{\alpha} = \varepsilon k(\Theta_{0,1}^{\alpha}Z - Z_{0,1}^{\alpha}\Theta) + s^{-1}\frac{d}{dr}(R_{0,1}^{\alpha}\Theta + \Theta_{0,1}^{\alpha}R) ,$$

$$Z^{\alpha}_{0,0} = s^{-1} \frac{1}{r} \frac{d}{dr} r(R^{\alpha}_{0,1} Z + Z^{\alpha}_{0,1} R) .$$

In minimizing δW it is only necessary (any other permissible harmonic vector will produce a positive contribution to δW in second order of ε) to consider

$$\vec{h}_{0,0} = \beta(R^{\beta}_{0,0}, i\Theta^{\beta}_{0,0}, iZ^{\beta}_{0,0}) \exp(i\phi' - i\phi) ,$$

where

$$R^{\beta}_{0,0} = (k'-k)^{-1} \frac{d}{dr} I_{\phi'-\phi} , \quad \Theta^{\beta}_{0,0} = 0 , \quad Z^{\beta}_{0,0} = I_{\phi'-\phi} .$$

Then, $\vec{\xi}_1$ satisfies the following equation.

$$(50) \qquad L\vec{\xi}_1 = \frac{1}{2} \vec{B}_0 \times \nabla \times \nabla \times \left[\vec{h}_{0,1} \times \vec{B}_1 + (\vec{h}_{0,0} + \vec{g}_{0,0}) \times (\vec{B}^*_1 + \vec{B}_1) \right] + O(\varepsilon) .$$

Observe that among various terms in $\vec{\xi}_2$ only those containing $\exp(i\phi')$ or $\exp(i\phi' - i\phi)$ will interact with $\vec{\xi}_0$ which contains $\exp(i\phi')$ and $\exp(i\phi' - i\phi)$ and contribute to δW in second order of ε. Thus, $\vec{\xi}_1$ must include terms containing $\exp(i\phi' + i\phi)$, $\exp(i\phi')$, $\exp(i\phi' - i\phi)$ or $\exp(i\phi' - i2\phi)$. Indeed it is only necessary to consider the following particular solution.

$$(51) \qquad \vec{\xi}_1 = \vec{f}_1 + \vec{g}_{1,2} + \vec{g}_{1,1} + \vec{g}_{1,-1} + \vec{h}_{1,2} + \vec{h}_{1,-1} .$$

Here $\vec{h}_{1,2}$ and $\vec{h}_{1,-1}$ are two harmonic vectors

$$\vec{h}_{1,2} = \gamma\left(R^{\gamma}_{1,2}, i\Theta^{\gamma}_{1,2}, iZ^{\gamma}_{1,2}\right) \exp(i\phi' + i\phi) ,$$

$$\vec{h}_{1,-1} = \delta\left(R^{\delta}_{1,-1}, i\Theta^{\delta}_{1,-1}, iZ^{\delta}_{i,-1}\right) \exp(i\phi' - i2\phi) ,$$

where

$$R^{\gamma}_{1,2} = \frac{d}{dr} I_{\phi'+\phi} , \quad \Theta^{\gamma}_{1,2} = \frac{m'+m}{r} I_{\phi'+\phi} , \quad Z^{\gamma}_{1,2} = (k'+k) I_{\phi'+\phi} ,$$

$$R^{\delta}_{1,-1} = \frac{d}{dr} I_{\phi'-2\phi}, \quad \Theta^{\delta}_{1,-1} = \frac{m'-2m}{r} I_{\phi'-2\phi}, \quad Z^{\delta}_{1,-1} = (k'-2k) I_{\phi'-2\phi} .$$

The vectors $\vec{g}_{1,2}$, $\vec{g}_{1,1}$ and $\vec{g}_{1,-1}$ satisfy

$$a \frac{\partial}{\partial z} \vec{g}_{1,2} + \frac{1}{2} \nabla\times(\vec{h}_{0,1}\times\vec{B}_1) = 0 , \quad \nabla(\nabla\cdot\vec{g}_{1,2}) = 0 ,$$

$$a \frac{\partial}{\partial z} \vec{g}_{1,1} + \frac{1}{2} \nabla\times \left[(\vec{h}_{0,0}+\vec{g}_{0,0})\times\vec{B}_1\right] = 0 , \quad \nabla(\nabla\cdot\vec{g}_{1,1}) = 0 ,$$

$$a \frac{\partial}{\partial z} \vec{g}_{1,-1} + \frac{1}{2} \nabla\times \left[(\vec{h}_{0,0}+\vec{g}_{0,0})\times\vec{B}^{*}_1\right] = 0 , \quad \nabla(\nabla\cdot\vec{g}_{1,-1}) = 0 ,$$

hence

$$\vec{g}_{1,2} = \frac{\alpha}{2} \left(R^{\alpha}_{1,2} , i\Theta^{\alpha}_{1,2} , iZ^{\alpha}_{1,2}\right) \exp(i\phi'+i\phi) ,$$

$$\vec{g}_{1,1} = \left[\frac{\alpha}{4} \left(R^{\alpha}_{1,1} , i\Theta^{\alpha}_{1,1} , iZ^{\alpha}_{1,1}\right) + \frac{\beta}{2}\left(R^{\beta}_{1,1} , i\Theta^{\beta}_{1,1} , iZ^{\beta}_{1,1}\right)\right] \exp(i\phi') ,$$

$$\vec{g}_{1,-1} = \left[\frac{\alpha}{4}\left(R^{\alpha}_{1,-1}, i\Theta^{\alpha}_{1,-1}, iZ^{\alpha}_{1,-1}\right) + \frac{\beta}{2}\left(R^{\beta}_{1,-1}, i\Theta^{\beta}_{1,-1}, iZ^{\beta}_{1,-1}\right)\right] \exp(i\phi'-i2\phi) ,$$

where

$$R^{\alpha}_{1,2} = \frac{k}{k'+k} \frac{m'+m}{r} (R^{\alpha}_{0,1}\Theta-\Theta^{\alpha}_{0,1}R) + k(R^{\alpha}_{0,1}Z-Z^{\alpha}_{0,1}R) = O(\varepsilon) ,$$

$$\Theta^{\alpha}_{1,2} = k(\Theta^{\alpha}_{0,1}Z-Z^{\alpha}_{0,1}\Theta) + \frac{k}{k'+k} \frac{d}{dr} (R^{\alpha}_{0,1}\Theta-\Theta^{\alpha}_{0,1}R) = O(\varepsilon) ,$$

$$Z^{\alpha}_{1,2} = \frac{k}{k'+k} \frac{1}{r} \frac{d}{dr} r(R^{\alpha}_{0,1}Z-Z^{\alpha}_{0,1}R) + \frac{k}{k'+k} \frac{m'+m}{r} (Z^{\alpha}_{0,1}\Theta-\Theta^{\alpha}_{0,1}Z) = O(\varepsilon) ,$$

$$R^{\alpha}_{1,1} = \frac{k}{k'} \frac{m'}{r} (R^{\alpha}_{0,0}\Theta-\Theta^{\alpha}_{0,0}R) + k(R^{\alpha}_{0,0}Z-Z^{\alpha}_{0,0}R) ,$$

$$\Theta^{\alpha}_{1,1} = k(\Theta^{\alpha}_{0,0}Z - Z^{\alpha}_{0,0}\Theta) + \frac{k}{k'}\frac{d}{dr}(R^{\alpha}_{0,0}\Theta - \Theta^{\alpha}_{0,0}R) ,$$

$$Z^{\alpha}_{1,1} = \frac{k}{k'}\frac{1}{r}\frac{d}{dr} r(R^{\alpha}_{0,0}Z - Z^{\alpha}_{0,0}R) + \frac{k}{k'}\frac{m'}{r}(Z^{\alpha}_{0,0}\Theta - \Theta^{\alpha}_{0,0}Z) ,$$

$$R^{\beta}_{1,1} = \frac{k}{k'}\frac{m'}{r} R^{\beta}_{0,0}\Theta + k(R^{\beta}_{0,0}Z - Z^{\beta}_{0,0}R) ,$$

$$\Theta^{\beta}_{1,1} = - k\ Z^{\beta}_{0,0}\Theta + \frac{k}{k'}\frac{d}{dr} R^{\beta}_{0,0}\Theta ,$$

$$Z^{\beta}_{1,1} = \frac{k}{k'}\frac{1}{r}\frac{d}{dr} r(R^{\beta}_{0,0}Z - Z^{\beta}_{0,0}R) + \frac{k}{k'}\frac{m'}{r} Z^{\beta}_{0,0}\Theta ,$$

$$R^{\alpha}_{1,-1} = \frac{k}{k'-2k}\frac{m'-2m}{r}(R^{\alpha}_{0,0}\Theta + \Theta^{\alpha}_{0,0}R) + k(R^{\alpha}_{0,0}Z + Z^{\alpha}_{0,0}R) ,$$

$$\Theta^{\alpha}_{1,-1} = k(\Theta^{\alpha}_{0,0}Z - Z^{\alpha}_{0,0}\Theta) + \frac{k}{k'-2k}\frac{d}{dr}(R^{\alpha}_{0,0}\Theta + \Theta^{\alpha}_{0,0}R) ,$$

$$Z^{\alpha}_{1,-1} = \frac{k}{k'-2k}\frac{1}{r}\frac{d}{dr} r(R^{\alpha}_{0,0}Z + Z^{\alpha}_{0,0}R) + \frac{k}{k'-2k}\frac{m'-2m}{r}(Z^{\alpha}_{0,0}\Theta - \Theta^{\alpha}_{0,0}Z) ,$$

$$R^{\beta}_{1,-1} = \frac{k}{k'-2k}\frac{m'-2m}{r}(R^{\beta}_{0,0}\Theta + \Theta^{\beta}_{0,0}R) + k(R^{\beta}_{0,0}Z + Z^{\beta}_{0,0}R) ,$$

$$\Theta^{\beta}_{1,-1} = k(\Theta^{\beta}_{0,0}Z - Z^{\beta}_{0,0}\Theta) + \frac{k}{k'-2k}\frac{d}{dr}(R^{\beta}_{0,0}\Theta - \Theta^{\beta}_{0,0}R) ,$$

$$Z^{\beta}_{1,-1} = \frac{k}{k'-2k}\frac{1}{r}\frac{d}{dr} r(R^{\beta}_{0,0}Z + Z^{\beta}_{0,0}R) + \frac{k}{k'-2k}\frac{m'-2m}{r}(Z^{\beta}_{0,0}\Theta - \Theta^{\beta}_{0,0}Z) .$$

And the undetermined vector $\vec{f}_1$, constrained by $\frac{\partial}{\partial z}\vec{f}_1 = O(\varepsilon)$ and $\nabla(\nabla\cdot\vec{f}_1) = O(\varepsilon)$, is to be determined by a solvability condition for the second order equation.

Since $\nabla\times\nabla\times(\vec{f}_0\times\vec{B}_0 + \frac{1}{2}\varepsilon\vec{h}_{0,1}\times\vec{B}_1^*) = 0$, $\nabla\times\nabla\times(\vec{g}_{1,2}\times\vec{B}_0 + \frac{1}{2}\vec{h}_{0,1}\times\vec{B}_1) = 0$, $\nabla\times\nabla\times(\vec{g}_{1,1}\times\vec{B}_0 + \frac{1}{2}\vec{f}_0\times\vec{B}_1) = 0$, and $\nabla\times\nabla\times(\vec{g}_{1,-1}\times\vec{B}_0 + \frac{1}{2}\vec{f}_0\times\vec{B}_1^*) = 0$, Eq. (22) becomes

$$\varepsilon^2 L\vec{\xi}_2 = -\varepsilon L\vec{f}_1 + \frac{\varepsilon^2}{2}\vec{B}_0\times\nabla\times\nabla\times\left|(\vec{f}_1+\vec{g}_{1,2}+\vec{g}_{1,1}+\vec{g}_{1,-1}+\vec{h}_{1,2}+\vec{h}_{1,-1})\right.$$

$$\left.\times(\vec{B}_1^*+\vec{B}_1) + (\vec{g}_{0,0}+\vec{h}_{0,1}+\vec{h}_{0,0})\times(\vec{B}_2^*+\vec{B}_2)\right| + O(\varepsilon^3) .$$

Its solvability requires that

$$-L\vec{f}_1 + \frac{\varepsilon}{2}\vec{B}_0\times\nabla\times\nabla\times\left[\vec{g}_{1,1}\times\vec{B}_1^* + (\vec{g}_{1,-1} + \vec{h}_{1,-1})\times\vec{B}_1\right] = O(\varepsilon^2) .$$

This is achieved provided that $\vec{f}_1 = \vec{g}_{1,0}$ satisfies

$$a\frac{\partial}{\partial z}\vec{g}_{1,0} + \frac{\varepsilon}{2}\nabla\times\left[\vec{g}_{1,1}\times\vec{B}_1^*+(\vec{g}_{1,-1}+\vec{h}_{1,-1})\times\vec{B}_1\right] = 0 , \quad \nabla(\nabla\cdot\vec{g}_{1,0}) = 0.$$

Hence,

$$\vec{g}_{1,0} = \left[\frac{\alpha}{8}(R^{\alpha}_{1,0}, i\Theta^{\alpha}_{1,0}, iZ^{\alpha}_{1,0}) + \frac{\beta}{4}(R^{\beta}_{1,0}, i\Theta^{\beta}_{1,0}, iZ^{\beta}_{1,0})\right.$$

$$\left. + \frac{\delta}{2}(R^{\delta}_{1,0}, i\Theta^{\delta}_{1,0}, iZ^{\delta}_{1,0})\right] \exp(i\phi'-i\phi) ,$$

where

$$R^{\alpha}_{1,0} = \varepsilon k\left[(R^{\alpha}_{1,1}+R^{\alpha}_{1,-1})Z + (Z^{\alpha}_{1,1}-Z^{\alpha}_{1,-1})R\right] ,$$

$$\Theta^{\alpha}_{1,0} = \varepsilon k\left[(\Theta^{\alpha}_{1,1}+\Theta^{\alpha}_{1,-1})Z-(Z^{\alpha}_{1,1}+Z^{\alpha}_{1,-1})\Theta\right]$$

$$+ s^{-1}\frac{d}{dr}\left[(R^{\alpha}_{1,1}+R^{\alpha}_{1,-1})\Theta+(\Theta^{\alpha}_{1,1}-\Theta^{\alpha}_{1,-1})R\right] ,$$

$$Z^{\alpha}_{1,0} = s^{-1}\frac{1}{r}\frac{d}{dr} r\left[(R^{\alpha}_{1,1}+R^{\alpha}_{1,-1})Z + (Z^{\alpha}_{1,1}-Z^{\alpha}_{1,-1})R\right] ,$$

$$R^{\beta}_{1,0} = \varepsilon k\left[(R^{\beta}_{1,1}+R^{\beta}_{1,-1})Z + (Z^{\beta}_{1,1}-Z^{\beta}_{1,-1})R\right] ,$$

$$\Theta^{\beta}_{1,0} = \varepsilon k\left[(\Theta^{\beta}_{1,1}+\Theta^{\beta}_{1,-1})Z-(Z^{\beta}_{1,1}+Z^{\beta}_{1,-1})\Theta\right]$$

$$+ s^{-1}\frac{d}{dr}\left[(R^{\beta}_{1,1}+R^{\beta}_{1,-1})\Theta+(\Theta^{\beta}_{1,1}-\Theta^{\beta}_{1,-1})R\right] ,$$

$$Z^{\beta}_{1,0} = s^{-1}\frac{1}{r}\frac{d}{dr} r\left[(R^{\beta}_{1,1}+R^{\beta}_{1,-1})Z + (Z^{\beta}_{1,1}-Z^{\beta}_{1,-1})R\right] ,$$

$$R^{\delta}_{1,0} = \varepsilon k(R^{\delta}_{1,-1}Z - Z^{\delta}_{1,-1}R) \ ,$$

$$\Theta^{\delta}_{1,0} = \varepsilon k(\Theta^{\delta}_{1,-1}Z - Z^{\delta}_{1,-1}\Theta) + s^{-1}\frac{d}{dr}(R^{\delta}_{1,-1}\Theta - \Theta^{\delta}_{1,-1}R) \ ,$$

$$Z^{\delta}_{1,0} = s^{-1}\frac{1}{r}\frac{d}{dr}\, r(R^{\delta}_{1,-1}Z - Z^{\delta}_{1,-1}R) \ .$$

Now $\vec{\xi}_2$ satisfies the following equation.

$$(52) \quad L\,\vec{\xi}_2 = \frac{1}{2}\vec{B}_0\times\nabla\times\nabla\times\Big[(\vec{g}_{1,2}+\vec{g}_{1,0}+\vec{g}_{1,-1}+\vec{h}_{1,2}+\vec{h}_{1,-1})\times\vec{B}^{*}_{1}$$

$$+ (\vec{g}_{1,2}+\vec{g}_{1,1}+\vec{g}_{1,0}+\vec{h}_{1,2})\times\vec{B}_1$$

$$+ (\vec{g}_{0,0}+\vec{h}_{0,1}+\vec{h}_{0,0})\times(\vec{B}^{*}_{2}+\vec{B}_2)\Big] + O(\varepsilon) \ .$$

The right side is a sum of five terms containing $\exp(i\phi'+i\nu\phi)$, $\nu = -3,-2,0,1,2$. Only the inhomogeneous solution corresponding to the term with $\exp(i\phi')$, which is denoted by $\vec{g}_{2,1}$ and satisfies

$$a\frac{\partial}{\partial z}\vec{g}_{2,1} + \frac{1}{2}\nabla\times\Big[(\vec{g}_{1,2}+\vec{h}_{1,2})\times\vec{B}^{*}_{1} + \vec{g}_{1,0}\times\vec{B}_1\Big] = 0, \quad \nabla(\nabla\cdot\vec{g}_{2,1}) = 0,$$

will contribute to δW in second order of ε. Thus,

$$(53) \quad \vec{g}_{2,1} = \Big[\frac{\alpha}{16}(R^{\alpha}_{2,1}, i\Theta^{\alpha}_{2,1}, iZ^{\alpha}_{2,1}) + \frac{\beta}{8}(R^{\beta}_{2,1}, i\Theta^{\beta}_{2,1}, iZ^{\beta}_{2,1})$$

$$+ \frac{\gamma}{2}(R^{\gamma}_{2,1}, i\Theta^{\gamma}_{2,1}, iZ^{\gamma}_{2,1}) + \frac{\delta}{4}(R^{\delta}_{2,1}, i\Theta^{\delta}_{2,1}, iZ^{\delta}_{2,1})\Big]\exp(i\phi'),$$

where

$R^{\alpha}_{2,1}, \Theta^{\alpha}_{2,1}, Z^{\alpha}_{2,1}, R^{\beta}_{2,1}, \Theta^{\beta}_{2,1}$ and $Z^{\beta}_{2,1}$ are $O(\varepsilon)$,

$$R^{\gamma}_{2,1} = \frac{k}{k'}\frac{m'}{r}(R^{\gamma}_{1,2}\Theta + \Theta^{\gamma}_{1,2}R) + k(R^{\gamma}_{1,2}Z + Z^{\gamma}_{1,2}R) \ ,$$

$$\Theta^{\gamma}_{2,1} = k(\Theta^{\gamma}_{1,2}Z - Z^{\gamma}_{1,2}\Theta) + \frac{k}{k'}\frac{d}{dr}(R^{\gamma}_{1,2}\Theta + \Theta^{\gamma}_{1,2}R) \ ,$$

$$Z^{\gamma}_{2,1} = \frac{k}{k'}\frac{1}{r}\frac{d}{dr}\, r(R^{\gamma}_{1,2}Z+Z^{\gamma}_{1,2}R) + \frac{k}{k'}\frac{m'}{r}(Z^{\gamma}_{1,2}\Theta-\Theta^{\gamma}_{1,2}Z) \ ,$$

$$R^{\delta}_{2,1} = \frac{k}{k'}\frac{m'}{r}(R^{\delta}_{1,0}\Theta-\Theta^{\delta}_{1,0}R) + k(R^{\delta}_{1,0}Z-Z^{\delta}_{1,0}R) \ ,$$

$$\Theta^{\delta}_{2,1} = k(\Theta^{\delta}_{1,0}Z - Z^{\delta}_{1,0}\Theta) + \frac{k}{k'}\frac{d}{dr}(R^{\delta}_{1,0}\Theta-\Theta^{\delta}_{1,0}R) \ ,$$

$$Z^{\delta}_{2,1} = \frac{k}{k'}\frac{1}{r}\frac{d}{dr}\, r(R^{\delta}_{1,0}Z-Z^{\delta}_{1,0}R) + \frac{k}{k'}\frac{m'}{r}(Z^{\delta}_{1,0}\Theta-\Theta^{\delta}_{1,0}Z) \ .$$

The displacement $\vec{\xi}$ just described is such that

(54) $$\nabla\cdot\vec{\xi} = O(\varepsilon^3) \ ,$$

and

$$a\frac{\partial}{\partial z}\vec{\xi} + \nabla\times\left\{\vec{\xi}\times\left|\frac{\varepsilon}{2}(\vec{B}_1^*+\vec{B}_1) + \frac{\varepsilon^2}{2}(\vec{B}_2^*+\vec{B}_2) + O(\varepsilon^3)\right|\right\}$$

$$= a\frac{\partial}{\partial z}\left|\vec{h}_{0,1}+\vec{h}_{0,0}+\varepsilon(\vec{h}_{1,2}+\vec{h}_{1,-1}) + O(\varepsilon^3)\right| \ .$$

Accordingly,

(55) $$\vec{Q} = a\frac{\partial}{\partial z}\left|\vec{h}_{0,1}+\vec{h}_{0,0}+\varepsilon(\vec{h}_{1,2}+\vec{h}_{1,-1}) + O(\varepsilon^3)\right|$$

$$= \alpha ak'\left(iR^{\alpha}_{0,1},-\Theta^{\alpha}_{0,1},-Z^{\alpha}_{0,1}\right)\exp(i\phi')$$

$$+ \beta a(k'-k)\left(iR^{\beta}_{0,0},-\Theta^{\beta}_{0,0},-Z^{\beta}_{0,0}\right)\exp(i\phi'-i\phi)$$

$$+ \varepsilon\gamma a(k'+k)\left(iR^{\gamma}_{1,2},-\Theta^{\gamma}_{1,2},-Z^{\gamma}_{1,2}\right)\exp(i\phi'+i\phi)$$

$$+ \varepsilon\delta a(k'-2k)\left(iR^{\delta}_{1,-1},-\Theta^{\delta}_{1,-1},-Z^{\delta}_{1,-1}\right)\exp(i\phi'-i2\phi) + O(\varepsilon^3) \ .$$

And

$$(56)\quad \vec{N}\cdot\nabla\times(\vec{\xi}\times\vec{B})\Big|_{r=\rho+0} = i\alpha E^{\alpha}_{0,1}\exp(i\phi')$$

$$+ i\varepsilon\Big[\Big(\frac{\alpha}{2} E^{\alpha}_{1,2}+\gamma E^{\gamma}_{1,2}\Big)\exp(i\phi'+i\phi) + \delta E^{\delta}_{1,-1}\exp(i\phi'-i2\phi)\Big]$$

$$+ i\varepsilon^2\Big[\Big(\frac{\alpha}{4} E^{\alpha}_{2,1}+\frac{\beta}{4} E^{\beta}_{2,1} + \frac{\gamma}{2} E^{\gamma}_{2,1}\Big)\exp(i\phi')$$

$$+ \Big(\frac{\alpha}{4} E^{\alpha}_{2,0} + \frac{\beta}{4} E^{\beta}_{2,0}\Big)\exp(i\phi'-i\phi) +\ldots\Big] + O(\varepsilon^3)\ .$$

The various coefficients, E's in (56) as well as F's in (57), A's in (58), B's in (59), and D's in (60) are given in Appendix D. The corresponding Φ in the vacuum region is

$$(57)\quad \Phi = i\alpha F^{\alpha}_{0,1}K_{\phi'}\exp(i\phi') + i\Big(\alpha F^{\alpha}_{0,0}+\beta F^{\beta}_{0,0}\Big)K_{\phi'-\phi}\exp(i\phi'-i\phi)$$

$$+ i\varepsilon\Big[\Big(\frac{\alpha}{2} F^{\alpha}_{1,2}+\gamma F^{\gamma}_{1,2}\Big)K_{\phi'+\phi}\exp(i\phi'+i\phi)+ \delta F^{\delta}_{1,-1}K_{\phi'-2\phi}\exp(i\phi'-i2\phi)\Big]$$

$$+ i\varepsilon^2\Big[\Big(\frac{\alpha}{4} F^{\alpha}_{2,1}+\frac{\beta}{2} F^{\beta}_{2,1} + \frac{\gamma}{2} F^{\gamma}_{2,1}\Big)K_{\phi'}\exp(i\phi')+ \ldots\Big] + O(\varepsilon^3)\ .$$

Now we compute the following.

$$(58)\quad (\vec{N}\cdot\vec{\xi})_{r=\rho} = \alpha A^{\alpha}_{0,1}\exp(i\phi')+ \varepsilon\Big[\Big(\frac{\alpha}{2} A^{\alpha}_{1,2}+\gamma A^{\gamma}_{1,2}\Big)\exp(i\phi'+i\phi)$$

$$+ \Big(\frac{\alpha}{2} A^{\alpha}_{1,0}+\beta A^{\beta}_{1,0}\Big)\exp(i\phi'-i\phi)+ \delta A^{\delta}_{1,-1}\exp(i\phi'-i2\phi)\Big]$$

$$+ \varepsilon^2\Big[\Big(\frac{\alpha}{4} A^{\alpha}_{2,1}+ \frac{\beta}{2} A^{\beta}_{2,1} + \frac{\gamma}{2} A^{\gamma}_{2,1}\Big)\exp(i\phi')+\ldots\Big] + O(\varepsilon^3)\ ,$$

$$(59)\quad (\Gamma p\nabla\cdot\vec{\xi}-\vec{B}\cdot\vec{Q})_{r=\rho} = \alpha B^{\alpha}_{0,1}\exp(i\phi') + \varepsilon\Big[\Big(\frac{\alpha}{2} B^{\alpha}_{1,2}+\gamma B^{\gamma}_{1,2}\Big)\exp(i\phi'+i\phi)$$

$$+ \Big(\frac{\alpha}{2} B^{\alpha}_{1,0}+\beta B^{\beta}_{1,0}\Big)\exp(i\phi'-i\phi) + \delta B^{\delta}_{1,-1}\exp(i\phi'-i2\phi)\Big]$$

$$+ \varepsilon^2\Big[\Big(\frac{\alpha}{4} B^{\alpha}_{2,1} + \frac{\beta}{2} B^{\beta}_{2,1} + \frac{\gamma}{2} B^{\gamma}_{2,1}\Big)\exp(i\phi')+\ldots\Big] + O(\varepsilon^3)\ ,$$

(60) $$(-\Phi)_{r=\rho} = i\alpha D^{\alpha}_{0,1}\exp(i\phi') + i\left(\alpha D^{\alpha}_{0,0}+\beta D^{\beta}_{0,0}\right)\exp(i\phi'-i\phi)$$

$$+ i\varepsilon\left[\left(\frac{\alpha}{2} D^{\alpha}_{1,2}+\gamma D^{\gamma}_{1,2}\right)\exp(i\phi'+i\phi)+ \delta D^{\delta}_{1,-1}\exp(i\phi'-i2\phi)\right]$$

$$+ i\varepsilon^2\left[\left(\frac{\alpha}{4} D^{\alpha}_{2,1}+ \frac{\beta}{2} D^{\beta}_{2,1} + \frac{\gamma}{2} D^{\gamma}_{2,1}\right)\exp(i\phi')+ \dots\right] + O(\varepsilon^3) ,$$

and $(\vec{N}\cdot\nabla\Phi)_{r=\rho}$ is given by (56).

Finally we obtain

(61) $$\delta W = \frac{\pi L}{2}\left[\alpha^*\alpha W(m',k')+\varepsilon^2\left(\frac{1}{4}\alpha^*\alpha W^{\alpha\alpha}+ \frac{1}{2}\alpha^*\beta W^{\alpha\beta}+ \frac{1}{2}\alpha^*\gamma W^{\alpha\gamma}\right.\right.$$

$$\left.\left. + \beta^*\beta W^{\beta\beta}+ \gamma^*\gamma W^{\gamma\gamma} + \delta^*\delta W^{\delta\delta}\right) + O(\varepsilon^3)\right] ,$$

where

$$W^{\alpha\alpha} = A^{\alpha}_{0,1}B^{\alpha}_{2,1} + A^{\alpha}_{1,0}B^{\alpha}_{1,0} + A^{\alpha}_{1,2}B^{\alpha}_{1,2} + A^{\alpha}_{2,1}B^{\alpha}_{0,1}$$

$$+ C_0(2A^{\alpha}_{0,1}A^{\alpha}_{2,1}+A^{\alpha}_{1,0}A^{\alpha}_{1,0}+A^{\alpha}_{1,2}A^{\alpha}_{1,2})$$

$$+ C_1A^{\alpha}_{0,1}(A^{\alpha}_{1,0}+A^{\alpha}_{1,2}) + C_2A^{\alpha}_{0,1}A^{\alpha}_{0,1}$$

$$+ D^{\alpha}_{0,1}E^{\alpha}_{2,1} + D^{\alpha}_{1,2}E^{\alpha}_{1,2} + D^{\alpha}_{2,1}E^{\alpha}_{0,1} + 2D^{\alpha}_{0,0}E^{\alpha}_{2,0} ,$$

$$W^{\alpha\beta} = A^{\alpha}_{0,1}B^{\beta}_{2,1} + A^{\alpha}_{1,0}B^{\beta}_{1,0} + B^{\alpha}_{1,0}A^{\beta}_{1,0}+B^{\alpha}_{0,1}A^{\beta}_{2,1}$$

$$+ 2C_0(A^{\alpha}_{0,1}A^{\beta}_{2,1}+A^{\alpha}_{1,0}A^{\beta}_{1,0}) + C_1A^{\alpha}_{0,1}A^{\beta}_{1,0}$$

$$+ D^{\alpha}_{0,1}E^{\beta}_{2,1} + E^{\alpha}_{0,1}D^{\beta}_{2,1} + 2D^{\alpha}_{0,0}E^{\beta}_{2,0} + E^{\alpha}_{2,0}D^{\beta}_{0,0} ,$$

$$W^{\alpha\gamma} = A^{\alpha}_{0,1}B^{\gamma}_{2,1}+A^{\alpha}_{1,2}B^{\gamma}_{1,2}+B^{\alpha}_{1,2}A^{\gamma}_{1,2}+B^{\alpha}_{0,1}A^{\gamma}_{2,1}+2C_0(A^{\alpha}_{0,1}A^{\gamma}_{2,1}+A^{\alpha}_{1,2}A^{\gamma}_{1,2})$$

$$+ C_1A^{\alpha}_{0,1}A^{\gamma}_{1,2}+ D^{\alpha}_{0,1}E^{\gamma}_{2,1}+ D^{\alpha}_{1,2}E^{\gamma}_{1,2}+ E^{\alpha}_{1,2}D^{\gamma}_{1,2}+ E^{\alpha}_{0,1}D^{\gamma}_{2,1} ,$$

$$W^{\beta\beta} = s^2k^2 \ W(m'-m,k'-k) = s^2k^2\left(2a^2-c^2+ \frac{2b^2}{e^2-1}\right) \quad ,$$

$$W^{\gamma\gamma} = W(m'+m,k'+k) \quad ,$$

$$W^{\delta\delta} = W(m'-2m,k'-2k) \quad .$$

Since k is chosen near the margin of stability, it differs only slightly from k_0, at which $\partial W/\partial k$ vanishes. Let $\frac{1}{2}(k'+k) - k_0 = \varepsilon ks'$. The value of $[W(1,k')-W(1,k_0)] - [W(1,k)-W(1,k_0)]$ gives $W(m',k') = \varepsilon^2 s'sk^2 \left.\frac{\partial^2 W}{\partial k^2}\right|_{k=k_0} + O(\varepsilon^3)$.

With $W^{\beta\beta} > 0$ and $W^{\gamma\gamma} > 0$, the minimum of (59) is

$$(62) \quad \delta W_{min} = \varepsilon^2 \frac{\pi L}{8} \alpha^*\alpha\left(W^{\alpha\alpha} - \frac{1}{4}\frac{W^{\alpha\beta}W^{\alpha\beta}}{W^{\beta\beta}} - \frac{1}{4}\frac{W^{\alpha\gamma}W^{\alpha\gamma}}{W^{\gamma\gamma}} + 4s'sk^2\frac{\partial^2 W}{\partial k^2}\right) + O(\varepsilon^3) \quad .$$

This is the minimum for δW when the disturbance is predominantly in the mode with $m' = 1$ and $k' = k(1+\varepsilon s)$.

We remark that the ratio of specific heats appears in (48) but not in (62). Indeed (62) does not reduce to (48) when s is set formally equal to zero. This nonuniform behavior is expected since the perturbation schemes required are quite different in the two cases. Presumably if an intermediate perturbation scheme is used to treat the gap for s greater than zero but smaller than first order in ε, a continuous variation in δW_{min} will emerge.

VI. Stable Equilibria with Long Helical Wavelength

The helical equilibrium in our discussion will be stable if both the expressions (48) and (62) for δW_{min} are positive. In this section we shall evaluate them in the limit of small kd, viz., when the helical structure has a long wavelength. This occurs when the axial fields are much greater than the azimuthal field, viz., $a^2 \gg c^2$, $b^2 \gg c^2$.

When the argument is small, the modified Bessel functions become

$$I_\nu(x) \approx 2^{-\nu}(\nu!)^{-1}x^\nu , \qquad K_\nu(x) \approx 2^{\nu-1}(\nu-1)!x^{-\nu} .$$

Thus

$$W(\nu,\kappa) = \frac{1}{\nu} a^2\kappa^2 - c^2 + \frac{1}{\nu}(\nu c+b\kappa)^2 \frac{e^{2\nu}+1}{e^{2\nu}-1} .$$

For small k and m = 1 we have approximately (evaluated at r = d = 1)

$$I_\phi = 1 , \quad I_{2\phi} = \frac{1}{2} ,$$

$$K_\phi = -\frac{e^2+1}{e^2-1} , \quad K_{2\phi} = -\frac{1}{2}\frac{e^4+1}{e^4-1} ,$$

$$I_{\phi'} = 1 , \quad I_{\phi'-\phi} = 2 , \quad I_{\phi'+\phi} = \frac{1}{2} ,$$

$$K_{\phi'} = -\frac{e^2+1}{e^2-1} , \quad K_{\phi'-\phi} = -\frac{2}{e^2-1} , \quad K_{\phi'+\phi} = -\frac{1}{2}\frac{e^4+1}{e^4-1} .$$

With the asymptotic values for the various constants A's, B's, C's, D's and E's as given in the ends of Appendices C and D of reference 10, we have

$$\lambda = \frac{-\frac{3}{2}c^2+4c(c+bk)\frac{e^4+e^2+1}{e^4-1} - 2(c+kb)^2\frac{e^6+2e^4+3e^2}{(e^2-1)(e^4-1)}}{2a^2k^2 - c^2 + 2(c+bk)^2\frac{e^4+1}{e^4-1}} ,$$

$$W^{\alpha\alpha} = 2a^2k^2+2c^2\frac{e^4+7}{e^4-1} - 4c(c+bk)\frac{e^6+e^4+15e^2-1}{(e^2-1)(e^4-1)}$$

$$+ 2(c+bk)^2 \frac{e^8+2e^6+30e^4-2e^2+1}{(e^2-1)^2(e^4-1)} ,$$

$$W'^{\alpha\alpha} = - 2a^2k^2 - c^2 \frac{3e^4-11}{e^4-1} + 4c(c+bk) \frac{3e^6-8e^2-3}{(e^2-1)(e^4-1)}$$

$$- 2(c+bk)^2 \frac{4e^6-2e^4-11e^2+1}{(e^2-1)^3}$$

$$+ \lambda \left[4a^2k^2 - 2c^2 - 8c(c+bk) \frac{e^4+e^2+1}{e^4-1} + 4(c+bk)^2 \frac{2e^4+2e^2-1}{(e^2-1)^2}\right],$$

$$W^{\alpha\beta} = \begin{cases} 3a^2k^2+c^2-4c(c+bk) \dfrac{e^2+1}{e^2-1} + (c+bk)^2 \dfrac{3e^4+4e^2-3}{(e^2-1)^2} , & \text{if } k'=k, \\ sk\left[6a^2k^2+2c^2 \dfrac{e^2+3}{e^2-1} - 8c(c+bk) \dfrac{e^4+3e^2-2e^2}{(e^2-1)^2} \right. & \\ \qquad \left. + 6(c+bk)^2 \dfrac{e^4+4e^2-1}{(e^2-1)^2}\right] , & \text{if } k'\neq k, \end{cases}$$

$$W^{\alpha\gamma} = 10a^2k^2-2c^2-16c(c+bk) \frac{e^4+e^2+1}{e^4-1} + 2(c+bk)^2 \frac{9e^6+9e^4+11e^2-5}{(e^2-1)(e^4-1)} ,$$

$$W^{\beta\beta} = \begin{cases} \frac{1}{2} (\Gamma p+a^2) - \frac{1}{4} c^2 , & \text{if } k' = k , \\ s^2(2a^2k^2 + \dfrac{2}{e^2-1} b^2k^2) , & \text{if } k' \neq k , \end{cases}$$

$$W^{\gamma\gamma} = 2a^2k^2 - c^2 + 2(c+bk)^2 \frac{e^4+1}{e^4-1} .$$

Therefore

$$(63) \quad \delta W_{min} = \varepsilon^2 \frac{\pi L}{8} \alpha^*\alpha \left[W^{\alpha\alpha} - \frac{1}{4} \frac{W^{\alpha\gamma}W^{\alpha\gamma}}{W^{\gamma\gamma}} - |W'^{\alpha\alpha}|\right] \quad \text{if } k' = k ,$$

$$(64) \quad \delta W_{min} = \varepsilon^2 \frac{\pi L}{8} \alpha^*\alpha \left[W^{\alpha\alpha} - \frac{1}{4} \frac{W^{\alpha\gamma}W^{\alpha\gamma}}{W^{\gamma\gamma}} + 8s's\left(a^2k^2+b^2k^2 \frac{e^2+1}{e^2-1}\right)\right] \text{if } k'\neq k.$$

Clearly if (63) is positive, then (64) can be made positive by proper restriction on the admissible values for k', which must be a multiple of $2\pi/L$. This is so when $2\pi/L$ is greater than the difference between the two roots of k for Eq. (9).

Now in the limit of small k, Eq. (9) becomes

$$\left(a^2+b^2 \frac{e^2+1}{e^2-1}\right)k^2 + 2bc \frac{e^2+1}{e^2-1} k + c^2 \frac{2}{e^2-1} = 0 .$$

Its two roots are

$$(65)\quad k = \frac{c}{b} \frac{e^2+1 \mp [(e^2-1)(e^2-1+2\beta_0)]^{1/2}}{(e^2-1)\beta_0-2e^2} ,$$

where $\beta_0 \equiv 1 - a^2/b^2$ is the plasma beta $[1-a^2/(b^2+c^2)+O(\varepsilon^2)]$ when $b^2 >> c^2$. The expression (63) contains three dimensionless parameters. They may be chosen as β_0, e, and c/b. Since c/b is regarded as small, there are effectively only two free parameters. For each set of values for β_0 and e, there are two values for k, corresponding to two different helical equilibria (with different values of L). In general, numerical evaluation is necessary to determine the algebraic sign of (63). However, some simplification is possible when β_0 is equal to zero or when e tends to unity. When $\beta_0 = 1$, the two roots for k are $-c/be^2$ and $-c/b$. With $k = -c/be^2$, (63) becomes $\varepsilon^2\alpha^*\alpha\pi L(e^2-2)^2/e^2(e^2-1)(e^4-3e^2+4)$, which is positive for all $e > 1$. On the other hand, with $k = -c/b$, (63) becomes $\varepsilon^2\alpha^*\alpha\pi L(-3e^4+7-|2e^4-4|)/2(e^4-1)$, which is positive for $1 < e^4 < 11/5$. When $e \to 1$, (63) becomes

$\varepsilon^2\alpha^*\alpha\pi L\beta_0(1-\beta_0)^{-1}(8-26\beta_0-|4-13\beta_0|)/16(e-1)^2$, which is positive for $0 < \beta_0 < 4/13$. Simplification also results when the two roots of k coalesce at $e^2-1+2\beta_0 = 0$. Then, the value of δW_{min},

$$\varepsilon^2\alpha^*\alpha\pi L \frac{\frac{1}{2}(e^2-1)(e^6+4e^4-e^2+8)-|e^6-3e^4+2e^2-2|}{(e^2+1)^2(e^6-e^4+7e^2+1)},$$

is positive for $e > 1.13$. But this special situation, namely, when the bifurcation point is exactly on the surface of marginal stability, does not have physical significance in the limit of small k, because the corresponding plasma beta is not positive.

In conclusion, there are stable helical equilibria. The shaded regions in Fig. 2 show the parameter ranges where the bifurcated equilibria with long helical wavelength are stable. The corresponding circular equilbria with the same magnetic fluxes and the same plasma volume are mostly unstable, as indicated by the undotted regions in Fig. 2.

Appendix A

In determining the marginal stability for a straight sheet pinch extensive numerical calculation [6,7,9] for the simultaneous algebraic equations $W(m,k) = 0$ and $dW/dk = 0$ were required. The laborious computations seem to have arisen from solving the transcendental equations for e/d and kd in terms of a/c and b/c. The difficulties can be obviated by instead solving the quadratic equations for a^2/c^2 and b^2/c^2 in terms of e/d and kd. Elimination of b yields [I and K below stand for $I(d;1,k)$ and $K(d;1,k)$ respectively]

$$\left(\frac{a}{c}\right)^4 k^6 \left(I \frac{dK}{dk} - K \frac{dI}{dk}\right)^2 - 2\left(\frac{a}{c}\right)^2 k^2 \left[2IK^3 + k\left(2K + k\frac{dK}{dk}\right)\left(I \frac{dK}{dk} - K\frac{dI}{dk}\right)\right] + 4K^3 + \left(2K + k\frac{dK}{dk}\right)^2 = 0$$

which possesses only one positive root for $(a/c)^2$, viz.,

$$\left(\frac{a}{c}\right)^2 = k^{-4}\left(I\frac{dK}{dk} - K\frac{dI}{dk}\right)^{-2} \left\{ 2IK^3 + k\left(2K + k\frac{dK}{dk}\right)\left(I\frac{dK}{dk} - K\frac{dI}{dk}\right) + 2K^2\left[I^2K^2 + k\left(2I + k\frac{dI}{dk}\right)\left(I\frac{dK}{dk} - K\frac{dI}{dk}\right)\right]^{1/2}\right\} .$$

Correspondingly,

$$\frac{b}{c} = k^{-2}\left(I\frac{dK}{dk} - K\frac{dI}{dk}\right)^{-1}\left\{ IK + \left[I^2K^2 + k\left(2I + k\frac{dI}{dk}\right)\left(I\frac{dK}{dk} - K\frac{dI}{dk}\right)\right]^{1/2}\right\} - k^{-1} .$$

The algebra simplifies in the following three asymptotic regimes:

(i) $e \to 1$: since $K \simeq -(1+k^2)^{-1}(e-1)^{-1}$ we have

$$a^2k^2I - c^2 \simeq 0 ,$$

$$c + bk \approx 0 .$$

Hence $a^2/c^2 \approx k^{-2}I^{-1}$ and $b/c \approx -k^{-1}$.

(ii) $k \to 0$: since $I \simeq 1$ and $K \approx -(e^2+1)(e^2-1)^{-1}$ we have

$$a^2k^2 - c^2 + (c+bk)^2(e^2+1)(e^2-1)^{-1} \approx 0 ,$$

$$a^2k^2 + bk(c+bk)(e^2+1)(e^2-1)^{-1} \approx 0 .$$

Hence $a^2/c^2 \approx 2k^{-2}(e^2+1)^{-1}$ and $b/c \approx -2k^{-1}(e^2+1)^{-1}$.

(iii) $k \to \infty$ [with $e-1 = O(k^{-1})$ and $k^2(e-1) << \cosh^2 k(e-1)$]: since $I \approx |k|^{-1}$ and $K \approx -k^{-1}\coth k(e-1)$ we have

$$a^2|k| - c^2 + (c+bk)^2 k^{-1} \coth k(e-1) \approx 0 ,$$

$$a^2|k| - 2c(c+bk)k^{-1} \coth k(e-1)$$

$$+ (c+bk)^2\left[k^{-1}\coth k(e-1) - (e-1)\operatorname{csch}^2 k(e-1)\right] \approx 0 .$$

Hence

$$\frac{a^2}{c^2} \approx \frac{1}{|k|}\left[1 - \frac{\sinh k(e-1)}{k(e-1)}\,\frac{\cosh^3 k(e-1)}{k^2(e-1)}\left\{1+\left[1+\frac{k^2(e-1)}{\cosh^2 k(e-1)}\right]^{1/2}\right\}^2\right],$$

and

$$\frac{b}{c} \approx -\frac{1}{k}\left[1 + \frac{\sinh 2k(e-1)}{2k(e-1)}\left\{1 + \left[1 + \frac{k^2(e-1)}{\cosh^2 k(e-1)}\right]^{1/2}\right\}\right] .$$

Figure 1 displays a^2/c^2 versus b^2/c^2 for various values of e/d with $|kd|$ as the varying parameter in each of these curves. The dotted curve connects the points on these solid curves where $2a^2-c^2+2b^2/(e^2-1)$ vanishes. The straight lines $a^2-(1-\beta)(b^2+c^2) = 0$ indicate the value of the plasma beta. It is seen that $\frac{e}{d}$ is in the range between 1 and 4.99 [both $W(1,k)$ and dW/dk vanish when $a^2/c^2 = 1$, $b^2/c^2 = 0$, $e/d = 4.99$ and $|kd| = 0.353$].

Appendix B

We shall solve the following equation for $\vec{x} \equiv (x_r, x_\theta, x_z)$.

$$L\,\vec{x} = 0 .$$

The linear partial differential operator L is defined by Eq. (31). From its three component equations

$$(\Gamma p + a^2)\frac{\partial}{\partial r}\nabla\cdot\vec{x} + a^2\left(\frac{\partial^2}{\partial x^2}x_r - \frac{\partial}{\partial r}\frac{\partial}{\partial z}x_z\right) = 0 ,$$

$$(\Gamma p + a^2)\frac{1}{r}\frac{\partial}{\partial\theta}\nabla\cdot\vec{x} + a^2\left(\frac{\partial^2}{\partial x^2}x_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial z}x_z\right) = 0 ,$$

$$\Gamma p\frac{\partial}{\partial z}\nabla\cdot\vec{x} = 0 ,$$

we find that

$$\frac{\partial^2}{\partial x^2}x_r - \frac{\partial}{\partial r}\frac{\partial}{\partial z}x_z = -\frac{\partial}{\partial r}g ,$$

$$\frac{\partial^2}{\partial z^2}x_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial z}x_z = -\frac{1}{r}\frac{\partial}{\partial\theta}g ,$$

$$\nabla\cdot\vec{x} = \frac{a^2}{\Gamma p + a^2}g ,$$

where $g(r,\theta)$ is an arbitrary function independent of z. Upon elimination of x_r and x_θ, we obtain

$$\nabla^2\frac{\partial}{\partial z}x_z = \nabla^2 g .$$

Hence

$$\frac{\partial}{\partial z}x_z = g + h + \frac{\partial^2 H}{\partial z^2} ,$$

where $h(x,\theta)$ is a harmonic function independent of z, and $H(r,\theta,z)$ is a harmonic function containing z. Accordingly,

$$\frac{\partial^2}{\partial z^2} x_r = \frac{\partial}{\partial r} (h + \frac{\partial^2 H}{\partial z^2}) \quad ,$$

$$\frac{\partial^2}{\partial z^2} x_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} (h + \frac{\partial^2 H}{\partial z^2}) \quad .$$

Integrations with respect to z twice yield

$$\vec{x} = \vec{f} + z(F,G,g+h) + \frac{1}{2} z^2 \nabla h + \nabla H \quad ,$$

where $\vec{f}(r,\theta)$, $F(r,\theta)$, and $G(r,\theta)$ are independent of x, but constrained by

$$\nabla \cdot \vec{f} + g + h = \frac{a^2}{\Gamma p + a^2} g \quad ,$$

$$\frac{1}{r} \frac{\partial}{\partial r} rF + \frac{1}{r} \frac{\partial}{\partial \theta} G = 0 \quad ,$$

in order to satisfy $\nabla \cdot \vec{x} = a^2 (\Gamma p + a^2)^{-1} g$. Now, periodicity in z requires that $F(r,\theta) = 0$, $G(r,\theta) = 0$, $g(r,\theta) + h(r,\theta) = 0$, and $h(r,\theta) = \text{const.}$

Therefore, the admissible solution of the Euler-Lagrange equation

$$L \vec{\xi} = 0$$

can be written as

$$\vec{\xi} = \sum_m \sum_{k \neq 0} \alpha_{m,k} \left(\frac{d}{dr} I_\phi \ , \ i \frac{m}{r} I_\phi \ , \ ikI_\phi \right) \exp(im\theta + ikz)$$

$$+ \sum_{m \neq 0} \beta_m \left(-i \frac{m}{r} F(r), \frac{d}{dr} F(r), G(r) \right) \exp(im\theta) + \beta \left(\frac{r}{2} , f(r), g(r) \right) ,$$

where F(r), G(r), f(r) and g(r) are arbitrary functions of r. The corresponding Φ satisfying Eqs. (26), (27), and (28) is

$$\Phi = \sum_m \sum_{k\neq 0} i\alpha_{m,k}\left(\frac{mc}{d} + kb\right) K_\phi \ \exp(im\theta + ikz)$$

$$+ \sum_{m\neq 0} \beta_m \frac{mc}{d} F(d) \left[\left(\frac{d}{e}\right)^m - \left(\frac{e}{d}\right)^m\right]^{-1} \left[\left(\frac{r}{e}\right)^m + \left(\frac{e}{r}\right)^m\right] \exp(im\theta) \ .$$

Accordingly,

$$\delta W = \frac{\pi L}{2} \left\{ \sum_m \sum_{k\neq 0} \alpha^*_{m,k}\alpha_{m,k} \ W(m,k)\, d^{-2} \right.$$

$$+ \sum_{m\neq 0} m^2 \beta^*_m \beta F(d) F(d) \left(- \frac{c^2}{d^2} + m \frac{c^2}{d^2} \frac{e^{2m}+d^{2m}}{e^{2m}-d^{2m}} \right)$$

$$\left. + \beta\beta \left[(\Gamma p + a^2) d^2 - \frac{1}{2} c^2 d^2 \right] \right\} \ .$$

It is seen that in this case various harmonic vectors are orthogonal in contributions to δW.

Appendix C

The various coefficients in (43), (44), (46), (40), and (41) are given below.

$$A^{\alpha}_{0,1} = 1 ,$$

$$A^{\alpha}_{1,2} = 2(m^2+k^2)I_{\phi} ,$$

$$A^{\alpha}_{1,0} = \frac{2a^2k^2}{\Gamma p+a^2} I_{\phi} ,$$

$$A^{\alpha}_{2,1} = R^{\alpha}_{2,1} - 2m^2 I_{\phi} + (m^2+k^2)(1+I_{\phi}) + \frac{2a^2k^2}{\Gamma p+a^2}\left[1+(m^2+k^2)I^2_{\phi}\right] ,$$

$$A^{\alpha}_{2,-1} = R^{\alpha}_{2,-1} + m^2 I_{\phi} - (m^2+k^2)\left(\frac{1}{2} + \frac{1}{2} I_{\phi} + \lambda I_{\phi}\right) + \frac{2a^2k^2}{\Gamma p+a^2}\left[1 + (m^2+k^2)I^2_{\phi}\right] ,$$

$$A^{\beta}_{1,0} = \frac{1}{2} ,$$

$$A^{\beta}_{2,1} = 1 + \frac{1}{2}(m^2+k^2)I_{\phi} ,$$

$$A^{\gamma}_{1,2} = 1 ,$$

$$A^{\gamma}_{2,1} = (m^2+k^2)(I_{\phi}+4I_{2\phi}) ;$$

$$B^{\alpha}_{0,1} = a^2k^2 I_{\phi} ,$$

$$B^{\alpha}_{1,2} = a^2k^2\left[2 - (m^2+k^2)I^2_{\phi}\right] ,$$

$$B^{\alpha}_{1,0} = - a^2k^2(m^2+k^2)I^2_{\phi} ,$$

$$B^{\alpha}_{2,1} = a^2k^2\left\{-1-(m^2+k^2)I_{\phi} + 2(m^2+k^2)^2 I^3_{\phi} + S_r - \frac{2a^2k^2}{\Gamma p+a^2} I_{\phi}\left[1 + (m^2+k^2)I^2_{\phi}\right]\right\} ,$$

$$B^{\alpha}_{2,-1} = a^2k^2\Big\{\frac{1}{2} - 2m^2 I^2_{\phi} + (m^2+k^2) I_{\phi}(\frac{1}{2} - 2I_{2\phi})$$

$$+ 2(m^2+k^2)^2 I^2_{\phi}(I_{\phi}+2I_{2\phi}) + S_r - \frac{2a^2k^2}{\Gamma p+a^2} I_{\phi}\left[1+(m^2+k^2)I^2_{\phi}\right]$$

$$- \lambda\left[1 + 4(m^2+k^2)I_{\phi}I_{2\phi}\right]\Big\} ,$$

$$B^{\beta}_{1,0} = \Gamma p+a^2 ,$$

$$B^{\beta}_{2,1} = - a^2k^2 I_{\phi} ,$$

$$B^{\gamma}_{1,2} = 4a^2k^2 I_{2\phi} ,$$

$$B^{\gamma}_{2,1} = a^2k^2\left[2 - 4(m^2+k^2)I_{\phi}I_{2\phi}\right] ;$$

$$D^{\alpha}_{0,1} = - E^{\alpha}_{0,1}K_{\phi} ,$$

$$D^{\alpha}_{1,2} = - E^{\alpha}_{1,2}K_{2\phi} + E^{\alpha}_{0,1}\left[-1 + 2(m^2+k^2)K_{\phi}K_{2\phi}\right] ,$$

$$D^{\alpha}_{1,0} = - E^{\alpha}_{0,1} ,$$

$$D^{\alpha}_{2,1} = - E^{\alpha}_{2,1}K_{\phi} + E^{\alpha}_{1,2}\left[-1 + 2(m^2+k^2)K_{\phi}K_{2\phi}\right]$$

$$+ E^{\alpha}_{0,1}\left[1-2m^2K^2_{\phi}+(m^2+k^2)(2K_{\phi}+K^2_{\phi})-4(m^2+k^2)^2K^2_{\phi}K_{2\phi}\right] ,$$

$$D^{\alpha}_{2,-1} = - E^{\alpha}_{2,-1}K_{\phi} + E^{\alpha}_{0,1}\Big\{\frac{1}{2} + m^2K^2_{\phi} - (m^2+k^2)(K_{\phi}+ \frac{1}{2}K^2_{\phi})$$

$$- \lambda\left[1 + (m^2+k^2)K^2_{\phi}\right]\Big\} ,$$

$$D^{\beta}_{2,1} = - E^{\beta}_{2,1}K_{\phi} ,$$

$$D^{\gamma}_{1,2} = - E^{\gamma}_{1,2}K_{2\phi} ,$$

$$D^{\gamma}_{2,1} = - E^{\gamma}_{2,1}K_{\phi} + E^{\gamma}_{1,2}\left[-1 + 2(m^2+k^2)K_{\phi}K_{2\phi}\right] ;$$

$$E^{\alpha}_{0,1} = mc + kb ,$$

$$E^{\alpha}_{1,2} = -4mc + (mc+kb)(m^2+k^2)(4I_{\phi}-2K_{\phi}) ,$$

$$E^{\alpha}_{2,1} = mc\left[6-4(m^2+k^2)I_{\phi}\right]+(mc+kb)\left[m^2(-2I_{\phi}+4K_{\phi})+(m^2+k^2)(-1+I_{\phi})\right.$$
$$\left.-2(m^2+k^2)^2 I_{\phi}K_{\phi}+R^{\alpha}_{2,1}\right] + \frac{2a^2k^2}{\Gamma p+a^2}\left\{-2mcI_{\phi}+(mc+kb)\left[1+\right.\right.$$
$$\left.\left.+(m^2+k^2)I_{\phi}(I_{\phi}-K_{\phi})\right]\right\} ,$$

$$E^{\alpha}_{2,-1} = -mc\left[3+4(m^2+k^2)K_{2\phi}\right] + (mc+kb)\left[-m^2(I_{\phi}+2K_{\phi})\right.$$
$$\left.+(m^2+k^2)(\tfrac{3}{2}+\tfrac{1}{2}I_{\phi}+2K_{2\phi})-4(m^2+k^2)^2K_{\phi}K_{2\phi}-R^{\alpha}_{2,-1}\right]$$
$$+\frac{2a^2k^2}{\Gamma p+a^2}\left\{2mcI_{\phi}-(mc+kb)\left[1+(m^2+k^2)I_{\phi}(I_{\phi}-K_{\phi})\right]\right\}$$
$$+\lambda\left[2mc+(mc+kb)(m^2+k^2)(I_{\phi}+4K_{2\phi})\right] ,$$

$$E^{\beta}_{2,1} = -mc + (mc+kb)\left[1+\tfrac{1}{2}(m^2+k^2)(I_{\phi}-K_{\phi})\right] ,$$

$$E^{\gamma}_{1,2} = 2(mc+kb) ,$$

$$E^{\gamma}_{2,1} = -2mc + (mc+kb)(m^2+k^2)(I_{\phi}-K_{\phi}+4I_{2\phi}) ;$$

$$F^{\alpha}_{0,1} = E^{\alpha}_{0,1} ,$$

$$F^{\alpha}_{1,2} = E^{\alpha}_{1,2} - 2E^{\alpha}_{0,1}(m^2+k^2)K_{\phi} ,$$

$$F^{\alpha}_{2,1} = E^{\alpha}_{2,1} - 2E^{\alpha}_{1,2}(m^2+k^2)K_{2\phi}+E^{\alpha}_{0,1}\left[2m^2K_{\phi}-(m^2+k^2)(1+K_{\phi})\right.$$
$$\left.+4(m^2+k^2)^2K_{\phi}K_{2\phi}\right] ,$$

$$F^{\alpha}_{2,-1} = E^{\alpha}_{2,-1} + E^{\alpha}_{0,1}\left[-m^2K_{\phi}+(m^2+k^2)(\tfrac{1}{2}+\tfrac{1}{2}K_{\phi}+\lambda K_{\phi})\right] ,$$

$$F^{\beta}_{2,1} = E^{\beta}_{2,1} ,$$

$$F^{\gamma}_{1,2} = E^{\gamma}_{1,2} ,$$

$$F^{\gamma}_{2,1} = E^{\gamma}_{2,1} - 2E^{\gamma}_{1,2}(m^2+k^2)K_{2\phi} .$$

In the limit of small k [a/c and b/c are $O(k^{-1})$] we have

$$\lambda \approx \frac{-\frac{3}{2}c^2+4c(c+bk)\dfrac{e^4+e^2+1}{e^4-1} - 2(c+bk)^2\dfrac{e^2(e^4+2e^2+3)}{(e^2-1)(e^4-1)}}{2a^2k^2 - c^2 + 2(c+bk)^2\dfrac{e^4+1}{e^4-1}} ;$$

$$A^{\alpha}_{0,1} = 1 ,$$

$$A^{\alpha}_{1,2} \approx 2 ,$$

$$A^{\alpha}_{1,0} \approx \frac{2a^2k^2}{\Gamma p+a^2} ,$$

$$A^{\alpha}_{2,1} = O(k^2) ,$$

$$A^{\alpha}_{2,-1} \approx -2 + 3\lambda + O(k^2) ,$$

$$A^{\beta}_{1,0} = \frac{1}{2} ,$$

$$A^{\beta}_{2,1} \approx \frac{3}{2} + O(k^2) ,$$

$$A^{\gamma}_{1,2} = 1 ,$$

$$A^{\gamma}_{2,1} \approx 3 ;$$

$$B^{\alpha}_{0,1} \approx a^2k^2 ,$$

$$B^{\alpha}_{1,2} \approx a^2k^2 ,$$

$$B^{\alpha}_{1,0} \approx -a^2k^2 ,$$

$$B^{\alpha}_{2,1} = O(k^2) ,$$

$$B^{\alpha}_{2,-1} \approx a^2k^2(2-3\lambda) ,$$

$$B^{\beta}_{1,0} = \Gamma p + a^2 ,$$

$$B^{\beta}_{2,1} \approx - a^2k^2 ,$$

$$B^{\gamma}_{1,2} \approx 2a^2k^2 ,$$

$$B^{\gamma}_{2,1} = - \frac{5}{6} a^2k^4 ;$$

$$C_0 = - c^2 ,$$

$$C_1 \approx 2a^2k^2 + 8c^2 - 4c(c+bk) \frac{e^2+1}{e^2-1} - 2(c+bk)^2 ,$$

$$C_2 \approx - 4 a^2k^2 - 18c^2 + 8c(c+bk) \frac{3e^2+1}{e^2-1} - 4(c+bk)^2 \frac{e^4+2e^2-1}{(e^2-1)^2} ,$$

$$\begin{aligned} C_3 \approx & - 2a^2k^2 - c^2 \frac{7e^4-15}{e^4-1} + 2c(c+bk) \frac{5e^6-e^4-11e^2-1}{(e^2-1)(e^4-1)} \\ & - 2(c+bk)^2 \frac{e^4+2e^2-1}{(e^2-1)^2} \\ & + 4\lambda\left[a^2k^2 + c^2 - c(c+bk) \frac{e^4+1}{e^4-1} - (c+bk)^2\right] ; \end{aligned}$$

$$D^{\alpha}_{0,1} \approx (c+bk) \frac{e^2+1}{e^2-1} ,$$

$$D^{\alpha}_{1,2} \approx - 2c \frac{e^4+1}{e^4-1} + (c+bk) \frac{3e^6+e^4+5e^2-1}{(e^2-1)(e^4-1)} ,$$

$$D^{\alpha}_{1,0} = - (c+bk) ,$$

$$D^{\alpha}_{2,1} \approx 2c \frac{e^4-4e^2-1}{(e^2-1)^2} - 4(c+bk) \frac{e^2(e^4-3e^2-2)}{(e^2-1)^3} ,$$

$$\begin{aligned} D^{\alpha}_{2,-1} \approx & - c \frac{e^4-5}{(e^2-1)^2} + 4(c+bk) \frac{e^2(e^4-e^2-2)}{(e^2-1)^3} \\ & + \lambda\left[2c \frac{e^2+1}{e^2-1} - (c+bk) \frac{7e^4+1}{(e^2-1)^2}\right] , \end{aligned}$$

$$D^{\beta}_{2,1} \approx -c\,\frac{e^2+1}{e^2-1} + (c+bk)\,\frac{(e^2+1)(2e^2-1)}{(e^2-1)^2}\,,$$

$$D^{\gamma}_{1,2} \approx (c+bk)\,\frac{e^4+1}{e^4-1}\,,$$

$$D^{\gamma}_{2,1} \approx -2c\,\frac{e^2+1}{e^2-1} + 2(c+bk)\,\frac{2e^4+3e^2-1}{(e^2-1)^2}\,;$$

$$E^{\alpha}_{0,1} = c + bk\,,$$

$$E^{\alpha}_{1,2} \approx -4c + 2(c+bk)\,\frac{3e^2-1}{e^2-1}\,,$$

$$E^{\alpha}_{2,1} \approx 2c - 4(c+bk)\,\frac{e^2}{e^2-1}\,,$$

$$E^{\alpha}_{2,-1} \approx -c\,\frac{e^4-5}{e^4-1} + 2(c+bk)\,\frac{e^2(e^4-e^2-4)}{(e^2-1)(e^4-1)} + \lambda\left[2c-(c+bk)\,\frac{5e^4-1}{e^4-1}\right]\,,$$

$$E^{\beta}_{2,1} \approx -c + (c+bk)\,\frac{2e^2-1}{e^2-1}\,,$$

$$E^{\gamma}_{1,2} = 2(c+bk)\,,$$

$$E^{\gamma}_{2,1} \approx -2c + 2(c+bk)\,\frac{2e^2-1}{e^2-1}\,.$$

Appendix D

The various coefficients in (58), (59), (60), (56), and (57) are given below.

$$A^{\alpha}_{0,1} = 1 ,$$

$$A^{\alpha}_{1,2} = \left[2 + k'(k'+k)\right] I_{\phi'} ,$$

$$A^{\alpha}_{1,0} = k^2 I_{\phi} + k'^2 I_{\phi'} ,$$

$$A^{\alpha}_{2,1} = 1 + k'^2 + k'k + k^2 + 2\frac{k}{k'} - 2\frac{k}{k'} I_{\phi} + (-1+k'^2 - 2\frac{k}{k'}) I_{\phi'}$$
$$+ k^2(\frac{k}{k'} + k^2) I_{\phi}^2 + \left[2k(k'+k) + k^2(k'^2+2k'k) + (2+k^2)\frac{k}{k'}\right] I_{\phi'} I_{\phi} ,$$

$$A^{\beta}_{1,0} = s\,k ,$$

$$A^{\beta}_{2,1} = sk\left[I_{\phi'-\phi} + (\frac{k'}{k} + k^2) I_{\phi}\right] ,$$

$$A^{\gamma}_{1,2} = 1 ,$$

$$A^{\gamma}_{2,1} = (\frac{k}{k'} + k^2) I_{\phi'} + \left[2(1+\frac{k}{k'}) + (k'+k)^2\right] I_{\phi'+\phi} ,$$

$$A^{\delta}_{1,-1} = 1 ;$$

$$B^{\alpha}_{0,1} = a^2 k'^2 I_{\phi'} ,$$

$$B^{\alpha}_{1,2} = a^2 k'^2 + a^2 k'k\left[1 - (1+k'k) I_{\phi'} I_{\phi}\right] ,$$

$$B^{\alpha}_{1,0} = a^2 k'^2 - a^2 k'k\left[1 + (1+k'k) I_{\phi'} I_{\phi}\right] ,$$

$$B^{\alpha}_{2,1} = a^2 k'^2\left[-1 + (1+k'^2) I_{\phi'}\right] + a^2 k'k\left[-2(1+k'k)(I_{\phi'} + I_{\phi}) + 4 I_{\phi'} I_{\phi}\right] ,$$

$$B^{\beta}_{1,0} = s a^2 k I_{\phi'-\phi} ,$$

$$B^{\beta}_{2,1} = sa^2k\left[k'(k'-k) - k^2 I_{\phi'-\phi} I_{\phi}\right] ,$$

$$B^{\gamma}_{1,2} = a^2(k'+k)^2 I_{\phi'+\phi} ,$$

$$B^{\gamma}_{2,1} = a^2(k'+k)\left[k' - k(2+k'k+k^2) I_{\phi} I_{\phi'+\phi}\right] ,$$

$$B^{\delta}_{1,-1} = a^2(k'-2k)^2 I_{\phi'-2\phi} ;$$

$$D^{\alpha}_{0,1} = - E^{\alpha}_{0,1} K_{\phi'} ,$$

$$D^{\alpha}_{0,0} = \frac{1}{2} s^{-1} K_{\phi'-\phi}\left[c(I_{\phi}+I_{\phi'}) + (c+bk)(-I_{\phi}+K_{\phi}) + (c+bk')(-I_{\phi'}+ \frac{k'}{k} K_{\phi'})\right],$$

$$D^{\alpha}_{1,2} = - E^{\alpha}_{1,2} K_{\phi'+\phi} + E^{\alpha}_{0,1}\left[-1 + (2+k'^2+k'k) K_{\phi'} K_{\phi'+\phi}\right] ,$$

$$D^{\alpha}_{2,1} = - E^{\alpha}_{2,1} K_{\phi'} + E^{\alpha}_{1,2}\left[-1 + (2+k'^2+k'k) K_{\phi'} K_{\phi'+\phi}\right]$$

$$+ E^{\alpha}_{0,1}\left[1+(2+k'^2+k'k) K_{\phi'} + (-1+k'^2) K^2_{\phi'} - (2+k'^2+k'k)^2 K^2_{\phi'} K_{\phi'+\phi}\right],$$

$$D^{\beta}_{0,0} = - bK_{\phi'-\phi} ,$$

$$D^{\beta}_{2,1} = skK_{\phi'}\left\{c(2-K_{\phi'-\phi}) + (c+bk)\left[-I_{\phi'-\phi}+(1+k'k) K_{\phi}\right]\right.$$

$$\left. + (c+bk')\left[-(\frac{k}{k'} +k^2) I_{\phi}+K_{\phi'-\phi}\right]\right\} ,$$

$$D^{\gamma}_{1,2} = - E^{\gamma}_{1,2} K_{\phi'+\phi} ,$$

$$D^{\gamma}_{2,1} = - E^{\gamma}_{2,1} K_{\phi'} + E^{\gamma}_{1,2}\left[-1 + (2+k'^2+k'k) K_{\phi'} K_{\phi'+\phi}\right] ,$$

$$D^{\delta}_{1,-1} = - E^{\delta}_{1,-1} K_{\phi'-2\phi} ;$$

$$E^{\alpha}_{0,1} = c + bk' ,$$

$$E^{\alpha}_{1,2} = - 4c + (c+bk')(1+k'^2) I_{\phi'} + (c+bk)\left[(3+2k'^2+k'k) I_{\phi'}\right.$$

$$\left. - (2+kk'+k^2) K_{\phi}\right] ,$$

$$E^{\alpha}_{2,1} = c\left[6 - 4(1+k'^2)I_{\phi'} - 2k(k'I_{\phi'}+kI_{\phi})\right]$$

$$+(c+bk')\left\{1+k'^2-(1-k'^2)I_{\phi'}+k^3I_{\phi}(k'I_{\phi'}+kI_{\phi})\right.$$

$$\left.+\frac{k}{k'}\left[2-2I_{\phi'}-2I_{\phi}+(2+k'^2+k^2)I_{\phi'}I_{\phi}+kI_{\phi}(k'I_{\phi'}+kI_{\phi})\right]\right\}$$

$$+(c+bk)\left\{-2-k'k+k^2+4K_{\phi}-k(1+k'k)K_{\phi}(k'I_{\phi'}+kI_{\phi})\right.$$

$$\left.+k\left[k'(1+k^2)+k(1+k'^2)\right]I_{\phi'}I_{\phi}-2(1+k'k)(1+k'^2)I_{\phi'}K_{\phi}\right\},$$

$$E^{\alpha}_{2,0} = sk^2\left[-cK_{\phi} + bk'I_{\phi'} + bk(I_{\phi}-K_{\phi})\right],$$

$$E^{\beta}_{2,1} = sk\left\{-2c + (c+bk')\left(\frac{k}{k'}+k^2\right)I_{\phi} + (c+bk)\left[I_{\phi'-\phi}-(1+k'k)K_{\phi}\right]\right\},$$

$$E^{\beta}_{2,0} = s^2bk^2,$$

$$E^{\gamma}_{1,2} = 2c + b(k'+k),$$

$$E^{\gamma}_{2,1} = -2c + (c+bk')\left[\left(\frac{k}{k'}+k^2\right)I_{\phi}+(k'+k)\left(\frac{2}{k'}+k'+k\right)I_{\phi'+\phi}\right]$$

$$-(c+bk)(1+k'k)K_{\phi},$$

$$E^{\delta}_{1,-1} = -c+b(k'-2k).$$

$$F^{\alpha}_{0,1} = E^{\alpha}_{0,1},$$

$$F^{\alpha}_{0,0} = \frac{1}{2}s^{-1}\left[b(kI_{\phi}+k'I_{\phi'}) - (c+bk)K_{\phi} - (c+bk')\frac{k'}{k}K_{\phi'}\right],$$

$$F^{\alpha}_{1,2} = E^{\alpha}_{1,2} - E^{\alpha}_{0,1}\left[2+k'(k'+k)\right]K_{\phi'},$$

$$F^{\alpha}_{2,1} = E^{\alpha}_{2,1} - E^{\alpha}_{1,2}\left[2+k'(k'+k)\right]K_{\phi'+\phi} - E^{\alpha}_{0,1}\left[1+k'^2+(-1+k'^2)K_{\phi'}\right.$$

$$\left.-(2+k'k'+k'k)^2K_{\phi'}K_{\phi'+\phi}\right] - 2sF^{\alpha}_{0,0}k'kK_{\phi'-\phi},$$

$$F^{\beta}_{0,0} = b \ ,$$

$$F^{\beta}_{2,1} = sk\left\{-2 + K_{\phi'-\phi} + (c+bk)\left[I_{\phi'-\phi}-(1+k'k)K_{\phi}\right]\right.$$

$$\left. + (c+bk')\left[(\frac{k}{k'} +k^2)I_{\phi}-K_{\phi'-\phi}\right]\right\} \ ,$$

$$F^{\gamma}_{1,2} = E^{\gamma}_{1,2} \ ,$$

$$F^{\gamma}_{2,1} = E^{\gamma}_{2,1} - E^{\gamma}_{1,2}\left[2 + k'(k'+k)\right]K_{\phi'+\phi} \ ,$$

$$F^{\delta}_{1,-1} = E^{\delta}_{1,-1} \ .$$

In the limit of small k [a/c and b/c are $O(k^{-1})$], we have

$$A^{\alpha}_{0,1} = 1 \ , \qquad B^{\alpha}_{0,1} \approx a^2k^2 \ ,$$

$$A^{\alpha}_{1,2} \approx 2 \ , \qquad B^{\alpha}_{1,2} \approx a^2k^2 \ ,$$

$$A^{\alpha}_{1,0} \approx 2k^2 \ , \qquad B^{\alpha}_{1,0} \approx - a^2k^2 \ ,$$

$$A^{\alpha}_{2,1} \approx \frac{41}{4}k^2 \ , \qquad B^{\alpha}_{2,1} \approx - \frac{17}{4}a^2k^4 \ ,$$

$$A^{\beta}_{1,0} = sk \ , \qquad B^{\beta}_{1,0} \approx 2sa^2k \ ,$$

$$A^{\beta}_{2,1} \approx 3sk \ , \qquad B^{\beta}_{2,1} \approx - 2sa^2k^3 \ ,$$

$$A^{\gamma}_{1,2} = 1 \ , \qquad B^{\gamma}_{1,2} \approx 2\ a^2k^2 \ ,$$

$$A^{\gamma}_{2,1} \approx 3 \ ; \qquad B^{\gamma}_{2,1} \approx - \frac{5}{6}a^2k^4 \ ;$$

$$D^{\alpha}_{0,1} \approx (c+bk)\,\frac{e^2+1}{e^2-1}\,,$$

$$D^{\alpha}_{0,0} \approx s^{-1}\,\frac{2}{e^2-1}\left[-c+2(c+bk)\,\frac{e^2}{e^2-1}\right],$$

$$D^{\alpha}_{1,2} \approx -\,2c\,\frac{e^4+1}{e^4-1} + (c+bk)\,\frac{3e^6+e^4+5e^2-1}{(e^2-1)(e^4-1)}\,,$$

$$D^{\alpha}_{2,1} \approx 2c\,\frac{e^4-4e^2-1}{(e^2-1)^2} - 4(c+bk)\,\frac{e^2(e^4-3e^2-2)}{(e^2-1)^3}\,,$$

$$D^{\beta}_{0,0} \approx b\,\frac{2}{e^2-1}\,,$$

$$D^{\beta}_{2,1} \approx 2sk\,\frac{e^2(e^2+1)}{(e^2-1)^2}\left[-c + 2(c+bk)\right],$$

$$D^{\gamma}_{1,2} \approx (c+bk)\,\frac{e^4+1}{e^4-1}\,,$$

$$D^{\gamma}_{2,1} \approx -\,2c\,\frac{e^2+1}{e^2-1} + 2(c+bk)\,\frac{2e^4+3e^2-1}{(e^2-1)^2}\,;$$

$$E^{\alpha}_{0,1} = c + bk\,,$$

$$E^{\alpha}_{1,2} \approx -\,4c + 2(c+bk)\,\frac{3e^2-1}{e^2-1}\,,$$

$$E^{\alpha}_{2,1} \approx 2c - 4(c+bk)\,\frac{e^2}{e^2-1}\,,$$

$$E^{\alpha}_{2,0} \approx sk^2\left[-2c + (c+bk)\,\frac{3e^2-1}{e^2-1}\right],$$

$$E^{\beta}_{2,1} \approx 2sk\left[-c + (c+bk)\,\frac{3e^2-1}{e^2-1}\right],$$

$$E^{\beta}_{2,0} \approx s^2b\,k^2\,,$$

$$E^{\gamma}_{1,2} = 2(c+bk)\,,$$

$$E^{\gamma}_{2,1} \approx -\,2c + 2(c+bk)\,\frac{2e^2-1}{e^2-1}\,.$$

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References

1. J. B. Keller and S. Antman (editors), Bifurcation Theory and Nonlinear Eigenvalue Problems. W. A. Benjamin, Inc. 1969.

2. K. O. Friedrichs, Rev. Modern Phys. 32, 889 (1960).

3. I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) A244, 17 (1957).

4. N. Friedman, Courant Institute of Mathematical Sciences Report NYO-1480-161, New York University, 1970.

5. H. Weitzner, Phys. Fluids 14, 658 (1971).

6. M. N. Rosenbluth, Los Alamos Scientific Laboratory Report No. 2030, 1956.

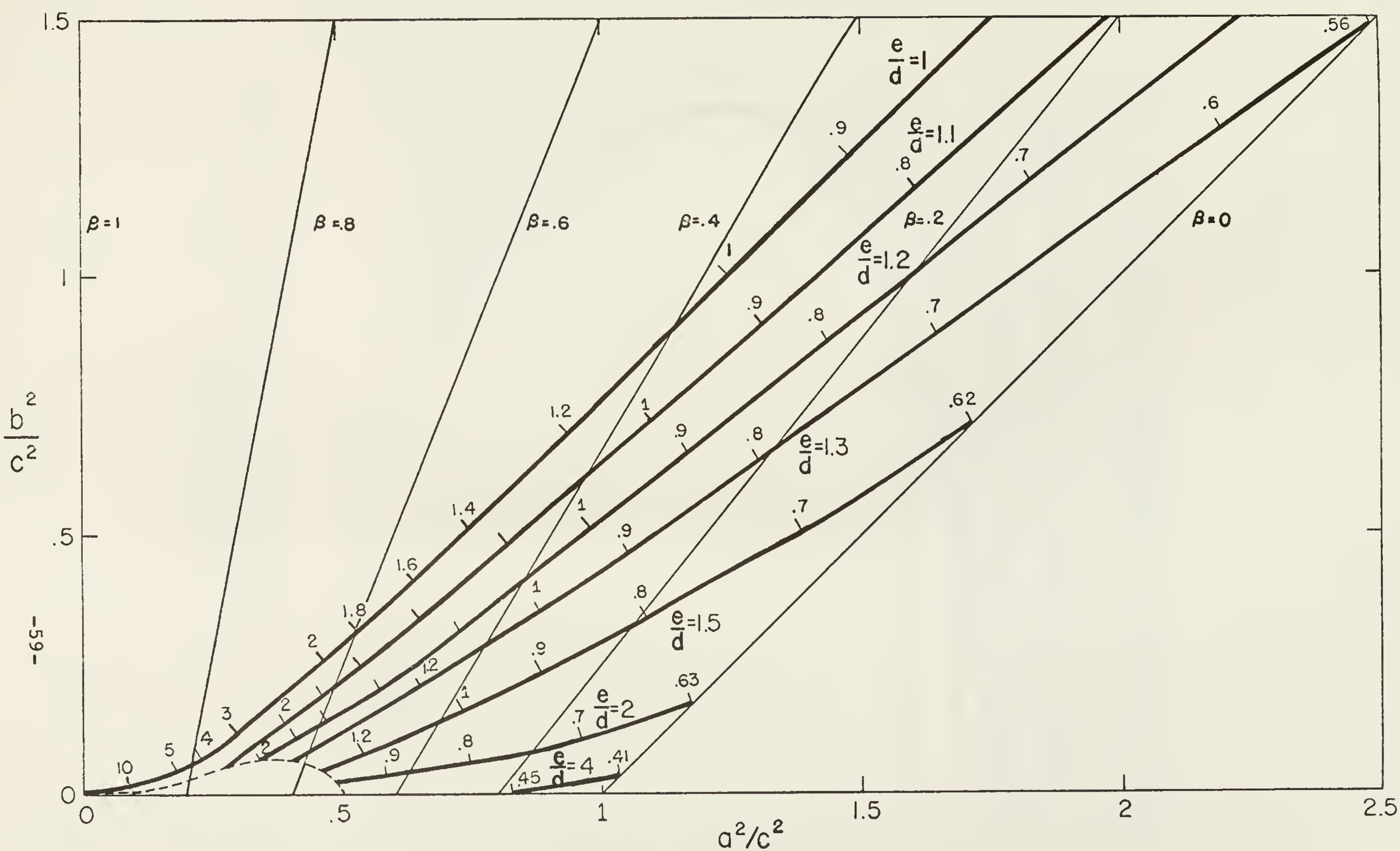
7. R. J. Tayler, Proc. Phys. Soc. (London) B70, 1049 (1957).

8. R. Lüst, B. R. Suydam, R. D. Richtmyer, A. Rotenberg, and D. Levy, Phys. Fluids 4, 891 (1961).

9. V. D. Shafranov, Plasma Physics and the Problem of Controlled Thermonuclear Reactions (ed. M. A. Leontovich), Vol. II, p. 197. Pergamon Press, 1959.

Figure 1

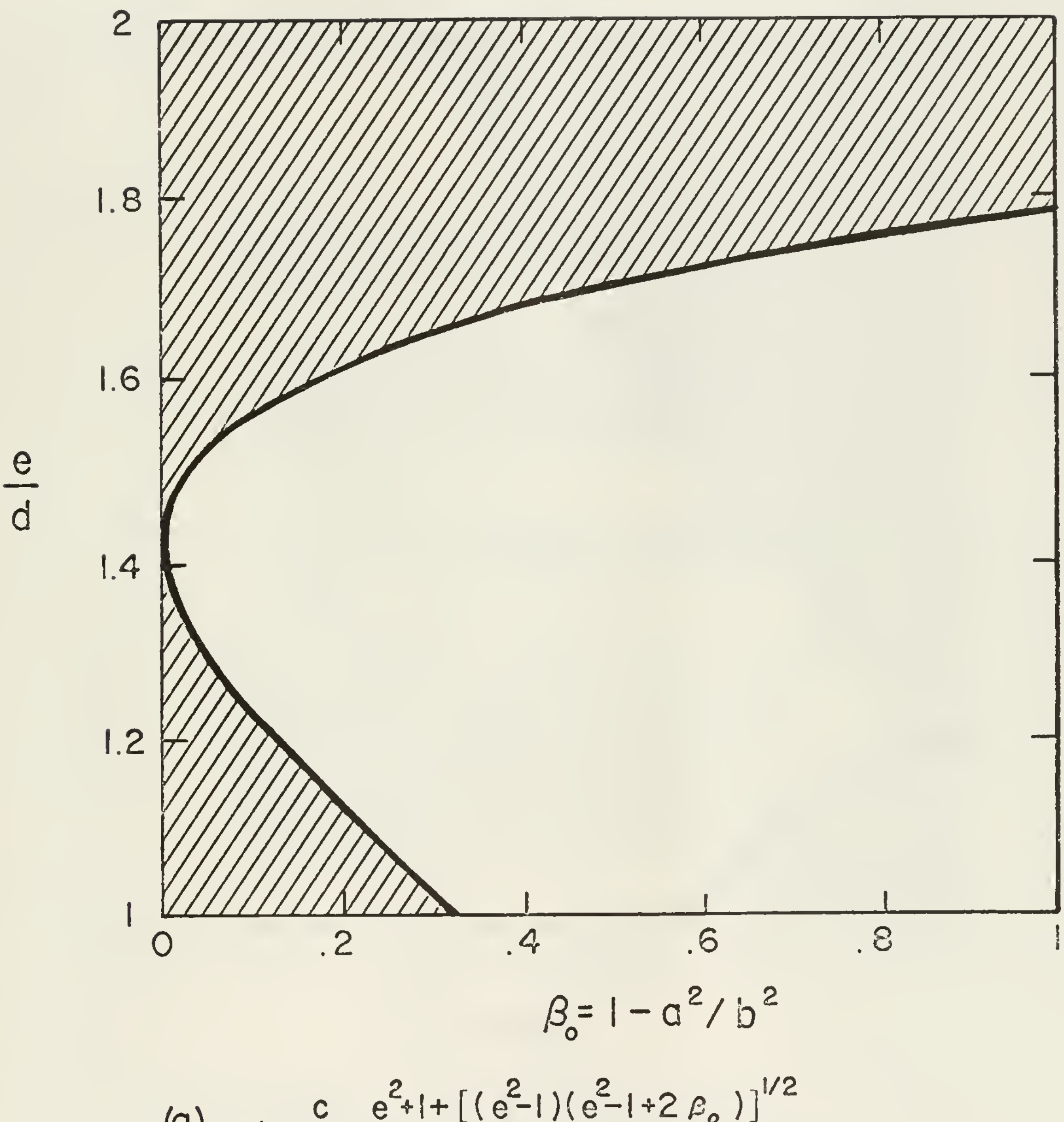
Solid curves (parameterized with $|kd|$) represent the surface of marginal stability for a straight sheet pinch in a parameter space for a^2/c^2, b^2/c^2 and e/d. The region above the surface is unstable for circular equilibria. The dotted curve represents the intersection of the surface of marginal stability with the surface $2a^2-c^2+2b^2(e^2/d^2-1)^{-1} = 0$. Thin lines indicate the value of the plasma beta $\beta = 1 - a^2/(b^2+c^2)$.



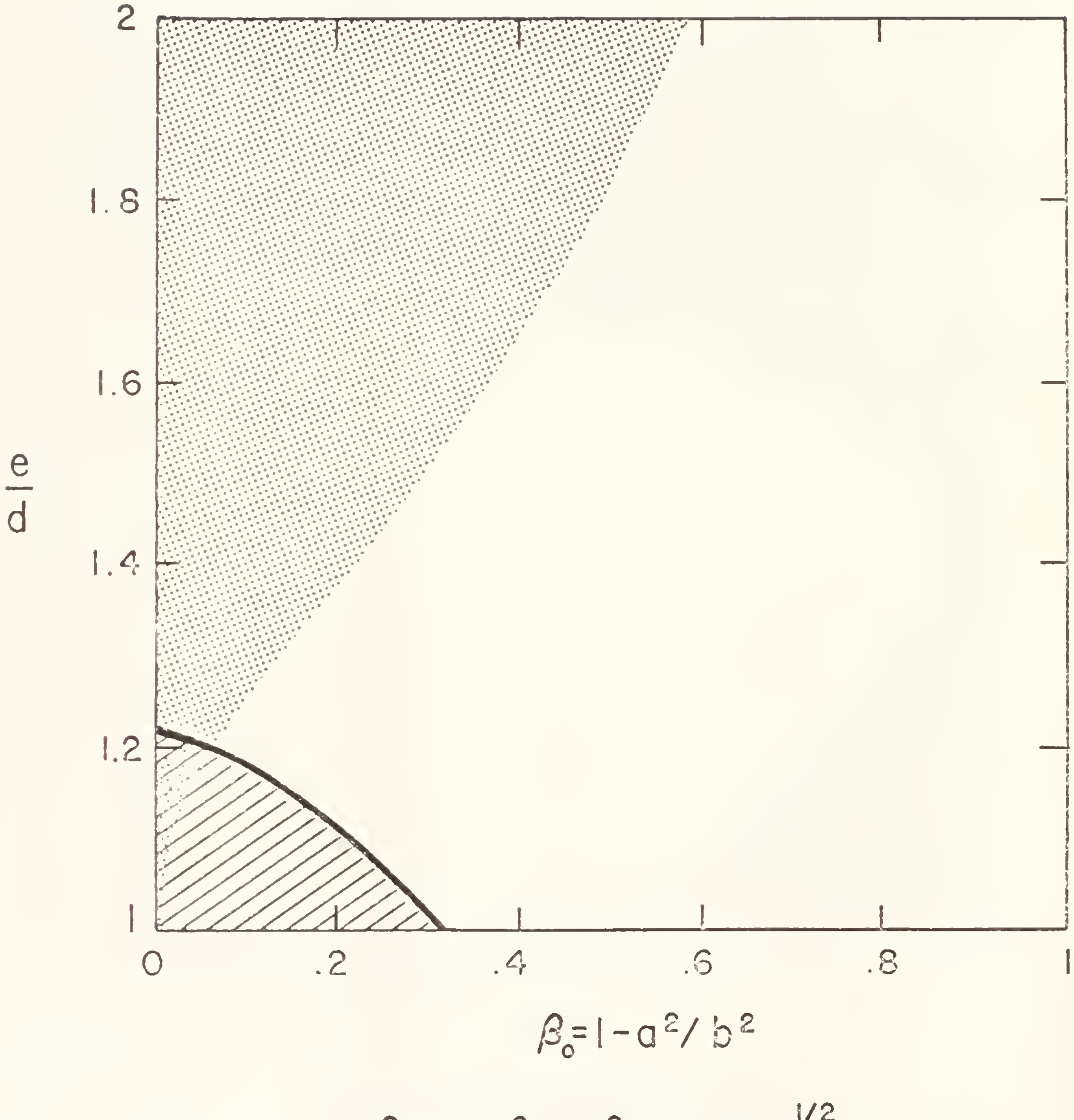
$\frac{b^2}{c^2}$
a^2/c^2
β=1
β=.8
β=.6
β=.4
β=.2
β=0
$\frac{e}{d}$=1
$\frac{e}{d}$=1.1
$\frac{e}{d}$=1.2
$\frac{e}{d}$=1.3
$\frac{e}{d}$=1.5
$\frac{e}{d}$=2
$\frac{e}{d}$=4
.56
.62
.63
.41
.45

Figure 2

Helical equilibria with long helical wavelength (kd << 1) are stable in the shaded regions. The corresponding circular equilibria are unstable in the undotted regions.



(a) $$k = \frac{c}{b} \quad \frac{e^2+1+[(e^2-1)(e^2-1+2\beta_o)]^{1/2}}{(e^2-1)\beta_o - 2e^2}$$



(b) $$k = \frac{c}{b}\,\frac{e^2+1+\left[(e^2-1)(e^2-1+2\beta_o)\right]^{1/2}}{(e^2-1)\beta_o - 2e^2}$$

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